MISSING OBSERVATIONS IN TIME SERIES AND THE «DUAL» AUTOCORRELATION FUNCTION

Agustín Maravall *
Daniel Peña **

* Banco de España.
** Universidad Politécnica de Madrid and University of Chicago. Daniel Peña acknowledges support from DGICYT, project number PB86-0538.

Banco de España. Servicio de Estudios
Documento de Trabajo n.º 8803

El Banco de España al publicar esta serie pretende facilitar la difusión de estudios de interés que contribuyan al mejor conocimiento de la economía española.

Los análisis, opiniones y conclusiones de estas investigaciones representan las ideas de los autores, con las que no necesariamente coincide el Banco de España.

ISBN: 84-7793-004-X
Depósito legal: M. 16841 - 1988

Imprenta del Banco de España

RESUMEN EN ESPAÑOL

Con cierta frecuencia se dispone de series incompletas, bien porque simplemente faltan datos en algunos períodos (aislados o en grupos), porque cambia la frecuencia de observación, o porque algunos de los datos son claramente erróneos. Aunque la literatura estadística se ha ocupado de ello desde hace algún tiempo, la estimación de las observaciones ausentes sigue siendo un problema computacional complicado. En los últimos años se ha extendido el algoritmo llamado EM para obtener estimadores óptimos de observaciones ausentes en series temporales.

El algoritmo EM es relativamente complejo y requiere el uso de rutinas de programación no-lineal y de filtro de Kalman, que no son fáciles de encontrar y manejar. De hecho, no existe ningún programa de uso general para la estimación de las observaciones que faltan en series incompletas.

En nuestro trabajo, relacionando el problema con el análisis de intervención y la extracción de señales, se obtiene inmediatamente la solución óptima, que puede expresarse en forma explícita de manera muy compacta. El análisis ofrece resultados adicionales de interés como es la expresión analítica del error cuadrático medio de las estimaciones, que también resulta muy compacta. Hemos obtenido la solución para el caso general de cualquier conjunto de secuencias de observaciones ausentes, colocadas en cualquier posición relativa de la serie temporal, y para cualquier serie temporal (representable por un filtro lineal). El análisis entronca de una manera natural con el tratamiento de observaciones atípicas e influyentes en series temporales.

Aunque los resultados tienen implicaciones computacionales de interés, en este primer trabajo solamente figuran los resultados teóricos, que resumimos a continuación.

Dado un modelo ARIMA, el modelo "dual o inverso" es el que resulta de intercambiar los operadores autorregresivo y de medias móviles. Así, por ejemplo, si el modelo se expresa en forma autorregresiva pura, como

$$\pi(B)\ z_t = a_t \quad ,$$

el modelo dual viene dado por

$$z_t = \pi(B)\ a_t \quad .$$

La varianza y la función de autocorrelación del modelo dual se obtienen de forma inmediata como

$$V = \sigma_a^2\ \Sigma\ \pi_i^2$$

$$\rho(B) = \pi(B)\ \pi(B^{-1})/\Sigma\ \pi_i^2 \quad .$$

En el presente trabajo demostramos que el estimador óptimo de una observación ausente en una serie temporal que sigue un modelo ARIMA es una media ponderada de las observaciones disponibles, donde los pesos son simplemente los coeficientes de la función de autocorrelación dual o inversa. Es decir, si la observación que falta es la correspondiente al período T, el estimador óptimo es igual a

$$\hat{z}_T = \Sigma_{i>0}\ \rho_i (z_{T+i} + z_{T-i}) \quad .$$

El error cuadrático medio del estimador es igual a la inversa de la varianza del proceso dual; es decir,

$$E(z_t - \hat{z}_T)^2 = 1/V \ .$$

El resultado se extiende de forma natural para el caso en que la observación que falta se encuentra cerca de los extremos de la serie y, a continuación, al caso en el que falta una secuencia de varias observaciones. En este último caso, el vector de estimadores óptimos se obtiene de la siguiente manera: se rellenan los "agujeros" en la serie con valores escogidos arbitrariamente. Se estima cada observación ausente mediante el método anterior, es decir, bajo el supuesto de que los demás valores arbitrarios son correctos, y el vector de estimadores que resulta se premultiplica por la inversa de la matriz de autocorrelación del modelo dual. Si el vector que resulta se sustrae del vector de valores arbitrarios, se obtienen los estimadores óptimos de las observaciones ausentes. El error cuadrático medio de estimación es la inversa de la matriz de autocorrelación dual. Finalmente, el resultado se generaliza fácilmente para cualquier conjunto de secuencias de cualquier longitud de observaciones ausentes.

SUMMARY

Given an ARIMA model, its "dual or inverse" model is the one that results from interchanging the autoregressive and moving average operators. In this paper, we prove that the optimal estimator of a missing observation in an ARIMA process is a weighted average of the observations, where the weights are the coefficients of the autocorrelation function of the dual process (i.e., the "dual or inverse" autocorrelation function); the mean squared estimation error is the inverse of the variance of the dual process. The result is extended to cover, first, the case of missing observations near the two extremes of the series; then, to the case of a sequence of missing observations, and finally, to the general case of any number of sequences of any length of missing observations. In all cases the optimal estimator can be expressed, in a compact way (trivial to compute), in terms of the dual autocorrelation function. The mean squared estimation error is always equal to the inverse of the (appropriately chosen) dual autocovariance matrix.

AMS 1980 subject classifications: 62M20, 62M10, 90A20.

Key words and phrases: Missing values, signal extraction, intervention analysis, influence analysis, ARIMA models, smoothing.

1. Introduction

In this paper we deal with estimation of missing observations in time series that are the outcome of Autoregressive Integrated Moving Average (ARIMA) models. We assume that the ARIMA model is known and hence we concern ourselves with obtaining the conditional expectation of the missing observation given the available ones, assuming the model is correct. We do not address the problem of re-estimation of the parameters of the model once the missing observations have been estimated. In terms of the well-known EM algorithm, we provide an answer to the E –the expectation– step. In a practical application, of course, if maximum likelihood restimation of the model yielded significantly different parameter estimates, new estimators of the missing observations should be computed with the new model, and the process could be iterated until convergence; see Peña (1987) or Little and Rubin (1987).

Thus, we assume that the series in question follows the general ARIMA model

$$\Phi(B)z_t = \Theta(B)a_t \ , \qquad (1.1)$$

where $\Phi(B)$ and $\Theta(B)$ are finite polynomials in the lag operator B, and a_t is a gaussian white-noise process (without loss of generality, we set $\sigma_a^2=1.$) The polinomial $\Phi(B)$ may contain any number of unit roots and hence the process may be nonstationary; we shall assume however, that the model is invertible, so that the roots of $\Theta(B)$ will lie outside the unit circle. The model (1.1) can alternatively be expressed as

$$z_t = \Psi(B)a_t \ , \qquad (1.2)$$

with $\Psi(B) = \Theta(B)/\Phi(B)$, or as

$$\pi(B)\, z_t = a_t \,, \qquad (1.3)$$

where $\pi(B)=\Psi(B)^{-1}=(1-\pi_1 B-\pi_2 B^2-\ldots)$. The "inverse or dual model" of an ARIMA model is the one that results from interchanging the AR and MA polynomials; therefore the dual model of (1.1) is given by

$$z_t^D = \pi(B) a_t \,, \qquad (1.4)$$

and the inverse or dual autocorrelation function (DACF) of z_t, first introduced by Cleveland (1972), is equal to

$$\rho^D(B) = \pi(B)\, \pi(F)/V_D \qquad (1.5)$$

where $F=B^{-1}$ and V_D is the variance of the inverse process, equal to

$$V_D = \Sigma_{i=0}^{\infty} \pi_i^2 \qquad (\pi_o=1) \quad .$$

Invertibility of a process guarantees stationarity of its dual process and the existence, thus, of the DACF. From the ARIMA expression of the model, the DACF is immediately available.

2. Optimal Smoothing of a Missing Value.

Consider a time series z_t with a missing value for $t=T$. Denote by $z_{(T)}$ the observed series $(\ldots, z_{T-2}, z_{T-1}, z_{T+1}, z_{T+2}, \ldots)$. The minimum mean-squared error (MMSE) estimator of z_T is given by

$$\hat{z}_T = E[z_T / z_{(T)}] \,,$$

that is

$$\hat{z}_T = \mathrm{Cov}(z_T, z_{(T)})' \, \mathrm{Var}^{-1}(z_{(T)}) z_{(T)} \quad ,$$

where $Cov(z_T, z_{(T)})$ is a vector of components $Cov(z_T z_i)$, $(i \neq T)$ and $Var(z_{(T)})$ is the covariance matrix of $z_{(T)}$. Therefore

$$\hat{z}_T = \Sigma_{i>0}\, \alpha_i\, (z_{T-i} + z_{T+i}) \quad ,$$

that is, z_T is a linear combination of the observed values, where the α_i weights depend on the covariance structure of the process. Several authors have shown how to compute $\hat{z}_T$ recursively using the Kalman filter (Jones, 1980; Harvey, 1981; Kohn and Ansley, 1983). Others have obtained the smoothing coefficients in particular cases (Abraham and Box, 1979; Miller and Ferreiro, 1984). A general expression, however, has not been derived. In order to do so, let us formulate the problem in the following manner:

Assume the series z_t is not observed and we observe instead Z_t, given by

$$\begin{aligned} Z_t &= z_t \quad , \qquad t \neq T \\ Z_T &= z_T + w \end{aligned} \tag{2.1}$$

where w is any unknown constant. Since Z_T is observed, if an appropriate estimator of w were available, then $\hat{z}_T$ could be computed through

$$\hat{z}_T = Z_T - \hat{w} \; . \tag{2.2}$$

In order to estimate w, define the dummy variable d_t, such that $d_t=0$ for $t \neq T$ and $d_T=1$, and write the intervention model

$$Z_t = w\, d_t + \Psi(B)\, a_t \; , \tag{2.3}$$

which is obtained by combining (1.2) and (2.1). The model (2.3) can alternatively be written

$$\pi(B)\ Z_t = w\ \pi(B)\ d_t + a_t\ ,$$

and, defining the variables $y_t = \pi(B)Z_T$ and $x_t = \pi(B)d_t$, it is seen to be the simple regression model

$$y_t = w\ x_t + a_t\ ,$$

with x_t deterministic and a_t white-noise. Assuming the observed series extends from t=1 to t=n, the optimal estimator of w is given by

$$\hat{w} = \Sigma\ y_t\ x_t / \Sigma\ x_t^2\ , \tag{2.4}$$

where all summation signs extend from t=1 to t=n. Assume $n\to\infty$; then, after simplification,

$$\Sigma\ y_t\ x_t = \Sigma\ \pi(B)\ Z_t\ \pi(B)\ d_t = \pi(B)\pi(F)\ Z_T$$

and

$$\Sigma\ x_t^2 = \Sigma\ \pi(B)\ d_t\ \pi(B)\ d_t = 1+\Sigma\ \pi_i^2 = V_D\ .$$

Therefore (2.4) becomes

$$\hat{w} = [\pi(B)\ \pi(F)/V_D]\ Z_T\ ,$$

or, according to (1.5) ,

$$\hat{w} = \rho^D(B)\ Z_T\ . \tag{2.5}$$

Introducing (2.5) in expression (2.2), the estimator of the missing observation z_T can be expressed as

$$\hat{z}_T = Z_T - \rho^D(B)\ Z_T = [1-\rho^D(B)]Z_T \tag{2.6}$$

Let ρ_k^D denote the coefficient of B^k (and of F^k) in $\rho^D(B)$. Since $\rho_o^D=1$ and $Z_t=z_t$ for $t\neq T$, the estimator $\hat{z}_T$ can be rewritten as

$$\hat{z}_T = - \Sigma_{k=1}^{\infty} \rho_k^D (z_{T+k} + z_{T-k}) \quad , \tag{2.7}$$

which tells us that the optimal estimator of the missing value is a symmetric linear combination of the observed values, where the weights are the coefficients of the dual autocorrelation function (centered at T). The filter (2.7) will be finite for a pure AR model and will extend to ∞ otherwise; invertibility of the model, however, guarantees its convergence in this last case.

If the process requires differencing the series (and hence is nonstationary,) $\pi(1)=0$ so that, from (1.5),

$$\rho^D(1) = 1 + 2\Sigma \rho_k^D = 0 \ ,$$

where the summation sign extends from 1 to ∞. Therefore $-\Sigma \rho_k^D=1/2$ and the sum of the weights in (2.7) is one; the estimator $\hat{z}_T$ is, in this case, a weighted mean of past and future values of the series. If the process is stationary, $\pi(1)>0$ from which it follows that

$$- \Sigma \rho_k^D = \frac{1}{2} \left[\frac{\Sigma \pi_i^2 - \pi(1)^2}{\Sigma \pi_i^2} \right] < \frac{1}{2} \quad ,$$

and hence the estimator $\hat{z}_T$ represents a shrinkage towards the mean of the process.

The result (2.7) provides a compact expression for the estimator, which can be easily implemented. As a first example, consider the random walk model

$$\nabla z_t = a_t \ .$$

The dual model is $z_t^D=(1-B)a_t$, and therefore $\rho_1^D=-1/2$, $\rho_k^D=0$ for $k=0,1$; thus

$$\hat{z}_T = (z_{T-1} + z_{T+1})/2 \quad .$$

This result justifies the practice of estimating missing observations in series of stock market prices or currency rates of exchange by simply averaging the two adjacent observations.

As a second example, for an AR(p) process it is easily found that

$$\rho_k^D = (-\phi_k + \Sigma_{i=1}^{p-k} \phi_i \ \phi_{i+k})/(1+\Sigma_{i=1}^{p} \phi_i^2) \ , \tag{2.8}$$

for k=1,...,p, and $\rho_k^D=0$ for k>p. Thus the optimal estimator is

$$\hat{z}_T = (1+\Sigma_{i=1}^{p} \phi_i^2)^{-1} \ \Sigma_{k=1}^{p} \ c_k \ (z_{T+k}+z_{T-k}) \ ,$$

where

$$c_k = \phi_k - \Sigma_{i=1}^{p-k} \phi_i \ \phi_{i+k} \quad ,$$

in agreement with the results of Abraham and Box (1979), Peña (1984) and Miller and Ferreiro (1984).

3. Smoothing and Signal Extraction

Since the estimator $\hat{z}_T$ does not depend on the observation z_T, (1.5) and (2.6) imply that it can be expressed as

$$\hat{z}_T = z_T - (1/V_D) \ \pi(B) \ \pi(F) \ z_T \ , \tag{3.1}$$

or, using (1.2),

$$\hat{z}_T = z_T - (1/V_D) \ \pi(F) \ a_T \ . \tag{3.2}$$

Therefore

$$\mathrm{Var}\ (\hat{z}_T) = \mathrm{Var}\ (z_T) + (1/V_D)^2\ \mathrm{Var}\ (\pi(F)\ a_T) -$$

$$- (2/V_D) E\ [\Psi(B)\ a_T\ \pi(F)\ a_T]\ .$$

Since $\mathrm{Var}(\pi(F)a_T) = V_D$ and the expectation at the end of the expression is equal to one,

$$\mathrm{Var}\ (\hat{z}_T) = \mathrm{Var}\ (z_T) - 1/V_D$$

so that the variance of the estimator is always smaller than that of the true observation. It is of interest to look at this smoothing from another angle.

Assume we wish to decompose the series z_t into signal plus noise, as in

$$z_t = s_t + u_t\ , \tag{3.3}$$

where $u_t \sim \mathrm{niid}(0,V_u)$ and s_t and u_t are mutually orthogonal. The MMSE estimator of the noise is given by (see, for example, Box, Hillmer and Tiao, 1978)

$$\hat{u}_t = [V_u/(\Psi(B)\ \Psi(F))]\ z_t$$

$$= V_u\ \pi(B)\ \pi(F)\ z_t\ . \tag{3.4}$$

and, considering (3.1), the estimator $\hat{z}_T$ can be expressed as

$$\hat{z}_T = z_T - 1/(V_D\ V_u)\ \hat{u}_T\ . \tag{3.5}$$

Using (1.2) in (3.4),

$$\hat{u}_t = V_u\ \pi(F)\ a_t\ ,$$

from which it is obtained that

$$\hat{V}_u = (V_u)^2 \, V_D \ , \tag{3.6}$$

where $\hat{V}_u$ denotes the variance of $\hat{u}_t$. Combining (3.5) and (3.6), the estimator $\hat{z}_T$ can then be written as

$$\hat{z}_T = z_T - k \, \hat{u}_T \tag{3.7}$$

where

$$k = V_u / \hat{V}_u \tag{3.8}$$

represents the ratio of the variances of the noise and of its MMSE estimator. Since the estimator has always a smaller variance (see, for example, Maravall, 1987), the ratio k is always greater than one. Thus the estimator of the missing observation is equal to the observation minus a multiple of the noise estimator. Given that Var $(k\hat{u}_t) = k^2 \hat{V}_u = k \, V_u$, what is substracted from the observation is also "larger" than the noise component of the series. The ratio k may vary considerably. To see two examples, for the random walk model k=2; for an Airline type of model with $\theta_1 = \theta_{12} = .25$ (see Box and Jenkins, 1976), k=25.5. Since, from (3.7), $\hat{z}_T$ can also be expressed as

$$\hat{z}_T = \hat{p}_T - (k-1)\hat{u}_t \quad ,$$

where (k–1)>0, the estimator of the missing observation is equal to the estimator of the signal that would be obtained with the complete series minus a multiple of the estimator of the noise.

4. Mean Squared Error of the Missing-Observation Estimator

From (3.7), $z_T - \hat{z}_T = k \, \hat{u}_t$, and hence

$$\text{MSE } (\hat{z}_T) = E(z_T-\hat{z}_T)^2 = k^2 \hat{V}_u \; .$$

From (3.6) and (3.8) it is obtained that $V_D = 1/(k^2 \hat{V}_u)$; thus

$$\text{MSE } (\hat{z}_T) = 1/V_D \; , \tag{4.1}$$

and the mean squared error of the estimator $\hat{z}_T$ is simply the inverse of the variance of the dual process, which is trivial to compute for any given model. Invertibility of the model implies that the MSE will always be finite and, since $V_D>1$, smaller than the one-period-ahead forecast variance, as should be expected.

5. An Alternative Interpretation of the Optimal Estimator

Consider the problem of estimating a missing observation at time T for a series that follows the AR(2) model

$$z_t = \Phi_1 \, z_{t-1} + \Phi_2 \, z_{t-2} + a_t \tag{5.1}$$

An obvious estimator of z_T is the one-period-ahead forecast of the series $[\ldots, z_{T-2}, z_{T-1}]$. Denoting this estimator by z_T^o,

$$z_T^o = \Phi_1 \, z_{T-1} + \Phi_2 \, z_{T-2} \; , \tag{5.2}$$

and the MSE of z_T^o, M_o is equal to $\sigma_a^2=1$.

Equation (5.2) is obtained by setting $a_t=0$ and t=T in (5.1); the estimator obtained obviously ignores the information z_{T+k}, k>0. An alternative estimator that uses this information can be obtained by considering z_T as the first element in the sequence $[z_T, z_{T+1}, z_{T+2}]$. This is equivalent to setting the innovation equal to zero and t=T+2 in (5.1), and the resulting equation can be solved to obtain the new estimator

$$z_T^2 = (z_{T+2} - \Phi_1 z_{T+1})/\Phi_2 \tag{5.3}$$

with associated MSE $M_2 = 1/\Phi_2^2$.

While z_T^o is computed as the final element of a series, z_T^2 is computed as the first element. Equation (5.1) still offers another possibility, namely, when z_T is in the middle. This will happen when t=T+1 in (5.1) and, solving for z_T, a third estimator is obtained:

$$z_T^1 = (z_{T+1} - \Phi_2 z_{T-1})/\Phi_1 \tag{5.4}$$

with MSE $M_1 = 1/\Phi_1^2$.

Finally, a "pooled" estimator of z_T can be obtained as a weighted average of the three previous estimators, where the weights are proportional to the precision of each estimator. If z_T^p denotes the pooled estimator,

$$z_T^p = h[z_T^o/M_o + z_T^1/M_1 + z_T^2/M_2] ,$$

where $h^{-1}=1/M_o+1/M_1+1/M_2$. Considering (5.2)-(5.4) and the values of M_o, M_1 and M_2, it is found that

$$z_T^p = [(\Phi_1-\Phi_1\Phi_2)(z_{T-1}+z_{T+1})+\Phi_2(z_{t-2}+z_{T+2})]/(1+\Phi_1^2+\Phi_2^2) ,$$

or, in view of (2.8),

$$z_T^p = -\rho_1^D(z_{T-1}+z_{T+1}) - \rho_2^D(z_{T-2}+z_{T+2}) .$$

Thus the pooled estimator is equal to the optimal estimator $\hat{z}_T$, derived in Section 2 and given by (2.7). Therefore, the optimal estimator of the missing observation can be seen as a weighted average of the estimators that are obtained by assuming that the missing observation occupies all possible different positions of z in the autoregressive equation (5.1).

The previous result for the AR(2) model generalizes to any linear invertible (possibly nonstationary) model of the type (1.1). To see this, write (1.1) as (1.3), i.e.

$$z_t = \pi_1 z_{t-1} + \pi_2 z_{t-2} + \dots + a_t , \tag{5.5}$$

or, for $t=T+j$, $(j = 0,1,2,\dots)$,

$$z_{T+j} = \pi_1 z_{T+j-1} + \pi_2 z_{T+j-2} + \dots + \pi_j z_T + \dots + a_{T+j} . \tag{5.6}$$

Setting $a_{T+j}=0$ and solving for z_T, the estimator z_T^j, given by

$$z_T^j = (1/\pi_j)(z_{T+j} - \pi_1 z_{T+j-1} - \dots) =$$

$$= (1/\pi_j)[\pi(B) z_{T+j} + \pi_j z_T] =$$

$$= (1/\pi_j)[\pi(B) F^j + \pi_j] z_T , \tag{5.7}$$

is obtained (for $j=0$ we adopt the convention $\pi_o=-1$.) The MSE of z_T^j is given by $M_j=1/\pi_j^2$. Letting $j=0,1,2,\dots$, the pooled estimator, z_T^p, is given by (all summation signs extend from $j=0$ to $j=\infty$)

$$z_T^p = h \Sigma z_T^j/M_j , \tag{5.8}$$

where $h^{-1} = \Sigma(1/M_j) = \Sigma \pi_j^2 = V_D$. Thus, using (5.7),

$$z_t^p = (1/V_D) \Sigma \pi_j [\pi(B)F^j + \pi_j] z_T =$$

$$= (1/V_D) [\Sigma \pi_j^2 z_T] + (1/V_D) \Sigma \pi_j F^j \pi(B) z_T =$$

$$= z_T - (1/V_D) \pi(B) \pi(F) z_T ,$$

and, considering (3.1), $z_T^p = \hat{z}_T$, as claimed.

6. Missing Observation Near the Two Extremes of the Series

6.1. Estimation

The optimal estimator of a missing observation at time T, derived in Section 2 and given by (2.7), is a symmetric filter centered at T. Although it extends theoretically from $-\infty$ to $+\infty$, invertibility of the series guarantees that the filter will converge towards zero, and hence that it can be truncated and applied to a finite length series. For T close enough to either end of the series, however, (2.7) cannot be used since observations needed to complete the filter will not be available.

Assume that the truncated filter extends from (T−N) to (T+N); that is, for k>N $\rho_k^D \simeq 0$. Let the available series consist of the (T+n−1) observations $[z_1, z_2, \ldots, z_{T-1}, z_{T+1}, \ldots, z_{T+n}]$. Two cases can be distinguished:

(A) If N>n, the "future" values $(z_{T+n+1}, \ldots, z_{T+N})$ are needed to compute $\hat{z}_T$ with (2.7), but they have not been observed yet.

(B) If 1>T−N, the "starting" values $(z_{T-N}, \ldots, z_o)$ are needed to compute $\hat{z}_T$, yet they are not available.

To simplify the discussion, assume that T+n>2N+1 (i.e., the length of the filter is smaller than the length of the series) so that cases (A) and (B) cannot occur simultaneously. Consider first case (A), when future observations would be needed in order to apply (2.7).

Similarly to the case of unobserved components estimation (such as, for example, seasonal adjustment) one can think of extending the series beyond z_{T+n} with forecasts, and apply the filter to the extended series (see, for example, Cleveland and Tiao, 1976). This procedure, however, cannot be used in the present context because of the following consideration: Since $\rho^D(B) = \pi(B)\pi(F)/V_D$, given that $n<N$, the fact that $\rho_k^D \neq 0$ for $k \leq N$ implies in general (not necessarily always) that $\pi_{n+1} \neq 0$. Consider the AR representation of the one-period-ahead forecast

$$\hat{z}_{T+n}(1) = \pi_1 z_{T+n} + \pi_2 z_{T+n-1} + \ldots + \pi_{n+1} z_T + \ldots$$

Since $\pi_{n+1} \neq 0$, it follows that the missing observation would be needed in order to compute the forecast.

To derive the optimal estimator of z_T when T is close to the end of the series, we use the method employed in section 4 to provide an alternative derivation of $\hat{z}_T$. From expression (5.5), only (n+1) equations of the type (5.6) can be obtained, namely those corresponding to $j=0,1,\ldots,n$, since z_{T+j} for $j>n$ has not been observed yet. Therefore, expression (5.8) remains valid with the summation sign extending now from $j=0$ to $j=n$, and $h^{-1} = \Sigma_{j=0}^{n} \pi_j^2$. Denote by V_D^n the truncated variance of the dual process,

$$V_D^n = \Sigma_{j=0}^{n} \pi_j^2 ,$$

and by $\pi^n(F)$ the truncated AR polynomial

$$\pi^n(F) = (1-\pi_1 F - \ldots - \pi_n F^n) .$$

then,

$$\hat{z}_T = (1/V_D^n) \Sigma_{j=o}^{n} \pi_j [\pi(B)F^j + \pi_j] z_T =$$

$$= z_T - (1/V_D^n) \pi(B)(\Sigma_{j=o}^{n} \pi_j F^j) z_T ,$$

or

$$\hat{z}_T = z_T - (1/V_D^n) \pi(B) \pi^n(F) z_T . \tag{6.1}$$

Since this expression does not depend on z_T, the estimator $\hat{z}_T$ can be written as

$$\hat{z}_T = Z_T - \hat{\omega} \; , \qquad (6.2a)$$

where

$$\hat{\omega} = (1/V_D^n) \; \pi(B) \; \pi^n(F) \; Z_T \; . \qquad (6.2b)$$

The equations in (6.2) provide an easy way of computing the optimal estimator of the missing observation. When n=0, (6.2) yields the one-period-ahead forecast of the series, while, when $n \to \infty$, (6.2) becomes the optimal estimator (2.7) for the case of an infinite series.

If the missing observation is near the beginning of the series –case (B)– the previous derivation would remain unchanged, applied to the "reversed" series $[z_{T+n} \ldots z_1]$. In this case expression (6.2a) holds, and (6.2b) becomes

$$\hat{\omega} = (1/V_D^n) \; \pi^n(B) \; \pi(F) \; Z_T \; .$$

6.2. Mean Squared Estimation Error.

When the last observation is for period (T+n), from (6.1) the error in the estimator of the missing observation is equal to

$$z_T - \hat{z}_T = (1/V_D^n) \; \pi(B) \; \pi^n(F) \; z_T \; .$$

Since $\pi(B) \; z_T = a_T$, this expression becomes

$$z_T - \hat{z}_T = (1/V_D^n) \; \pi^n(F) \; a_T \; ,$$

and considering that $E[\pi^n \, (F) \; a_T]^2 = V_D^n$, the MSE of the estimator is equal to

$$MSE\ (\hat{z}_T) = 1/V_D^n \quad , \tag{6.3}$$

the inverse of the truncated variance of the dual process, an expression which is trivial to compute given the model (1.1). Of course, $MSE(\hat{z}_t)$ reaches a maximum for n=0 (in which case it is equal to $\sigma_a^2 = 1$), and a minimum for $n\to\infty$, when $V_D^n \to V_D$.

7. A Sequence of Missing Observations

7.1. Estimation

Consider, first, a time series z_t with two missing observations at t=T and t=T+1. Using an approach similar to that in Section 2, we construct the observed series Z_t, given by:

$$Z_t = z_t \ , \qquad (t\neq T,\ T+1)$$

$$Z_T = z_T + \omega_o \ ,$$

$$Z_{T+1} = z_{T+1} + \omega_1 \ ,$$

where ω_o and ω_1 are any unknown constants. In order to estimate these constants, define the dummy variables

$$d_t^o = 0 \quad , \qquad t\neq T \ ,$$

$$d_T^o = 1$$

and

$$d_t^1 = 0 \quad , \qquad t\neq T+1 \ ,$$

$$d_{T+1}^1 = 1 \ ,$$

and, the intervention model

$$Z_t = \omega_o\, d_t^o + \omega_1\, d_t^1 + \Psi(B)\, a_t$$

can be writen as

$$\pi(B)\, Z_t = \omega_o\, \pi(B)\, d_t^o + \omega_1\, \pi(B)\, d_t^1 + a_t\ . \qquad (7.1)$$

Defining the variables

$$y_t = \pi(B)\, Z_t; \quad x_{ot} = \pi(B)\, d_t^o; \quad x_{1t} = \pi(B)\, d_t^1\ ,$$

expression (7.1) becomes

$$y_t = \omega_o\, x_{ot} + \omega_1\, x_{1t} + a_t\ .$$

The estimator of ω_o is then given by

$$\hat{\omega}_o = \frac{\Sigma x_{1t}^2\ \ \Sigma x_{ot}\, y_t - \Sigma x_{ot}\, x_{1t}\ \ \Sigma x_{1t}\, y_t}{\Sigma x_{ot}^2\ \ \Sigma x_{1t}^2\ - (\Sigma x_{ot}\, x_{1t})^2}\ . \qquad (7.2)$$

Letting $n \to \infty$, straightforward computation yields (summation signs extend from 1 to ∞)

$$\Sigma x_{ot}\, y_t = \pi(B)\, \pi(F)\, Z_T$$

$$\Sigma x_{1t}\, y_t = \pi(B)\, \pi(F)\, Z_{T+1}$$

$$\Sigma x_{ot}^2 = \Sigma x_{1t}^2 = V_D$$

$$\Sigma x_{ot}\, x_{1t} = \gamma_1^D\ ,$$

where $V_D = 1 + \Sigma \pi_j^2$ and $\gamma_1^D = -\pi_1 + \Sigma \pi_j\, \pi_{j+1}$ are the variance and lag-one autocovariance of the dual process. Replacing the above expressions in (7.2), it is obtained that

$$\hat{\omega}_o = \frac{V_D\ \pi(B)\pi(F)\ Z_T - \gamma_1^D \pi(B)\pi(F)\ Z_{T+1}}{V_D^2 - (\gamma_1^D)^2} =$$

$$= \frac{V_D^{-1}\ \pi(B)\pi(F)\ Z_T - \rho_1^D V_D^{-1} \pi(B)\pi(F)\ Z_{T+1}}{1 - (\rho_1^D)^2} =$$

$$= \frac{\rho^D\ (B)\ Z_T - \rho_1^D\ \rho^D(B)\ Z_{T+1}}{1 - (\rho_1^D)^2}\ . \qquad (7.3)$$

Let $\omega_o^{(1)}$ represent the estimator of ω_o obtained by assuming that the only missing value is z_T (i.e., by setting ω_1=0,) and let $\omega_1^{(1)}$ represent the estimator of ω_1 obtained by assuming that the the only missing value is z_{T+1} (i.e., by setting ω_o=0.) Then, from (2.5),

$$\omega_j^{(1)} = \rho^D(B)\ Z_{T+j} \qquad (j=0,1)\ ,$$

and (7.3) can be rewriten:

$$\hat{\omega}_o = \frac{\omega_o^{(1)} - \rho_1^D\ \omega_1^{(1)}}{1-(\rho_1^D)^2}\ , \qquad (7.4)$$

so that the estimator of the missing observation z_T can be obtained through

$$\hat{z}_T = Z_T - \hat{\omega}_o\ . \qquad (7.5)$$

When the two observations z_T and z_{T+1} are missing, expressions (7.4) and (7.5) offer an easy procedure to estimate them.

First, construct a new series Z_t, such that $Z_t = z_t$ for $t \neq T$, $T+1$, and Z_t is equal to any arbitrary number for $t=T$ and $T+1$. Next, estimate ω_o treating Z_T as the only missing value; this yields $\omega_o^{(1)}$. Similarly, estimate ω_1 treating Z_{T+1} as the only missing value; this yields $\omega_1^{(1)}$. Compute then $\hat{\omega}_o$ as the linear combination of $\omega_o^{(1)}$ and $\omega_1^{(1)}$ given by (7.4). Substracting $\hat{\omega}_o$ from (the arbitrarily chosen) Z_T, the estimator $\hat{z}_T$ is obtained.

The estimator $\hat{\omega}_o$, given by (7.4), is a linear combination of the observed series Z_t. Since the coefficient of Z_T in $\rho^D(B)\, Z_T$ is one, and in $\rho^D(B)\, Z_{T+1}$ it is ρ_1^D, it follows that the coefficient of Z_T in (7.4) is one. Furthermore, since the coefficient of Z_{T+1} in $\rho^D(B)\, Z_T$ is ρ_1^D, and in $\rho^D(B)\, Z_{T+1}$ it is one, the coefficient of Z_{T+1} in (7.4) is zero. Considering (7.5), the estimator z_t does not depend then on (Z_T, Z_{T+1}), the two arbitrary numbers.

In order to obtain an estimator of the second missing observation, $\hat{z}_{T+1}$, proceeding in a similar fashion we find

$$\hat{\omega}_1 = \frac{\omega_1^{(1)} - \rho_1^D\, \omega_o^{(1)}}{1 - (\rho_1^D)^2} \tag{7.6}$$

so that, again, $\hat{z}_{T+1}$, does not depend on (Z_T, Z_{T+1}). It is easily seen that expresions (7.4) and (7.6) can be written jointly as

$$\begin{pmatrix} \omega_o \\ \omega_1 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1^D \\ \rho_1^D & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega_o^{(1)} \\ \omega_1^{(1)} \end{pmatrix} .$$

The generalization to the case of k consecutive missing observations $(z_T, \ldots, z_{T+K-1})$ is straightforward: Let for the rest of the section, $j=0,1,\ldots,k-1$. The observed series can be written as

$$Z_t = z_t \qquad t \notin (T,\ldots,T+k-1)$$

$$Z_{T+j} = z_{T+j} + \omega_j \ .$$

The set of dummy variables are given by

$$d_t^j = 0 \qquad \text{for } t \neq T+j$$

$$d_{T+j}^j = 1 \ .$$

The regression equation becomes

$$y_t = \Sigma_j \ \omega_j \ x_{jt} + a_t \ , \tag{7.7}$$

where

$$y_t = \pi(B) \ Z_t$$

$$x_{jt} = \pi(B) \ d_t^j \ .$$

Let $\hat{\omega}$ denote the vector $(\hat{\omega}_o \ldots \hat{\omega}_{k-1})'$, X_j the column vector with element $[x_{jt}]$, and X the matrix $X=[X_o, X_1 \ldots X_{k-1}]$. Then, from (7.7),

$$\hat{\omega} = (X'X)^{-1} \ X'y \ . \tag{7.8}$$

Since (summing over t)

$$\Sigma x_{jt} \ y_t = \pi(B) \ \pi(F) \ Z_{t+j} \tag{7.9a}$$

$$\Sigma x_{jt}^2 = V_D \tag{7.9b}$$

$$\Sigma x_{jt} \ x_{j+h,t} = -\pi_h + \Sigma_{i=1}^{\infty} \ \pi_i \ \pi_{i+h} = \gamma_h^D \ , \tag{7.9c}$$

where γ_h^D denotes the lag-h autocovariance of the dual process, using (7.9b and c) the matrix (X'X) has all the elements of the j-th diagonal equal to γ_j^D $(V_D=\gamma_o^D)$. Therefore X'X can be written as the symmetric matrix:

$$X'X = V_D \begin{pmatrix} 1 & \rho_1^D & \cdots & \rho_{k-1}^D \\ & \cdot & & \\ & & \cdot & \rho_1^D \\ & & & 1 \end{pmatrix} = V_D \, R_D = \Omega_D \tag{7.10}$$

where R_D and Ω_D are the autocorrelation and autocovariance matrices of the dual process, truncated to be of order (kxk). Using (7.9a) and (7.10), (7.8) can be expressed as

$$\hat{\omega} = R_D^{-1} \, V_D^{-1} \, \pi(B)\pi(F) \begin{pmatrix} Z_T \\ \cdot \\ \cdot \\ Z_{T+K-1} \end{pmatrix} = R_D^{-1} \, \rho^D \, (B)Z$$

where Z denotes the vector $(Z_T \ldots Z_{T+k-1})'$, or, considering (2.5),

$$\hat{\omega} = R_D^{-1} \, \omega^{(1)} \, , \tag{7.11}$$

where $\omega^{(1)}$ is the vector $(\omega_o^{(1)} \ldots \omega_{k-1}^{(1)})'$, and $\omega_j^{(1)}$, as before, denotes the estimator of ω_j obtained by assuming that Z_{T+j} is the only missing observation (i.e., setting all other ω's equal to zero.) Then, the missing observations estimators can be computed through

$$\hat{z}_{T+j} = Z_{T+j} - \hat{\omega}_j \, , \tag{7.12}$$

and it can be shown that they do not depend on the arbitrary numbers $Z_T \ldots Z_{T+k-1}$.

Each estimator in (7.11) is a linear combination of the individual estimators that are obtained by assuming, for each missing observation, that the arbitrarily chosen numbers for the other missing observations are true, and applying the method of Section 2. The weights are the rows of the inverse of the dual autocorrelation matrix (truncated to be of the same order as the number of missing observations.)

7.2. Mean-Squared Estimation Error

Since the MSE of $\hat{\omega}$ in (7.8) is the matrix $(X'X)^{-1}$, using (7.10) and noticing that

$$z_{T+j} - \hat{z}_{T+j} = \hat{\omega}_j - \omega_j \quad ,$$

it follows that the MSE of the estimator $\hat{z} = (\hat{z}_T, \ldots, \hat{z}_{T+k-1})'$ is given by

$$\text{MSE}(\hat{z}) = (V_D \; R_D)^{-1} = \Omega_D^{-1}$$

where Ω_D is the autocovariance matrix of the dual process.

8. The General Case

We have seen how to estimate an isolated missing observation or a sequence of consecutive missing observations. The method of Section 7 can be easily extended to cover the general case of any arbitrary mixture of missing observations, whereby some may be isolated, some may be consecutive, and their relative distances in the series may not be large enough to allow for separate estimation.

Assume that, in general, the series z_t has k missing observations at periods T, $T+m_1$, $T+m_2, \ldots,$ $T+m_{k-1}$, where $m_1, \ldots, m_{k-1}$ are positive integers such that $m_1 < m_2 \ldots < m_{k-1}$. Let $j=0,1,\ldots,k-1$, and define the dummy variables:

$$d_t^j = 0 \quad \text{for} \quad t \neq T+m_j$$

$$= 1 \quad \text{"} \quad t = T+m_j \quad ,$$

where $m_o=0$. The regression equation is given by (7.7), where y_t and x_{jt} are as before and the vector $\hat{\omega}=(\omega_o \ldots \omega_{k-1})'$ is also given by (7.8). The expressions in (7.9) remain unchanged, except for (7.9a) which becomes now

$$\Sigma x_{jt}\, y_t = \pi(B)\ \pi(F)\ Z_{T+m_j} \quad .$$

The matrix (X'X) is equal to

$$X'X = V_D\ R$$

where R is the (kxk) symmetric matrix:

$$R = \begin{pmatrix} 1 & \rho^D_{m_1} & \rho^D_{m_2} \cdots & \rho^D_{m_{k-1}} \\ & 1 & \rho^D_{m_2-m_1} \cdots & \rho^D_{m_{k-1}-m_1} \\ & & 1 & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \\ & & & 1 \end{pmatrix} \qquad (8.1)$$

and therefore

$$\hat{\omega} = R^{-1}\ \rho^D(B) \begin{pmatrix} Z_T \\ Z_{T+m_1} \\ \cdot \\ \cdot \\ Z_{T+m_{k-1}} \end{pmatrix}$$

or

$$\hat{\omega} = R^{-1} \omega^{(1)} ,$$

and the missing observations estimator can be obtained through

$$\hat{z} = Z - \hat{\omega} .$$

Estimation of the missing observations in the general case amounts, thus, to the following procedure: First, fill the holes in the series with arbitrary numbers. Compute then $\omega_o^{(1)}$ as in Section 2 (i.e., assuming only the observation for t=T is missing.) In a similar way, compute $\omega_j^{(1)}$ and hence the vector $\omega^{(1)}$. Form the matrix R given by (8.1). Then $R^{-1} \omega^{(1)}$ yields $\hat{\omega}$ and, substracting this vector from the arbitrarily chosen vector Z, the vector $\hat{z}$ of estimators of the missing observations is obtained. Using the same derivation as in Section 7.2, the MSE matrix of the estimator $\hat{z}$ is given by the matrix $(V_D R)^{-1}$, or

$$MSE(\hat{z}) = \begin{pmatrix} 1 & \gamma^D_{m_1} & \gamma^D_{m_2} \cdots & \gamma^D_{m_{k-1}} \\ & 1 & \gamma^D_{m_2-m_1} \cdots & \gamma^D_{m_{k-1}-m_1} \\ & & 1 & \\ & & & \ddots \\ & & & 1 \end{pmatrix}^{-1} ,$$

where γ^D_j is the j-th order autocovariance of the dual process.

As an example, assume the series z_t has missing observations for t=T, T+1 and T+4. The matrix R is then equal to

$$R = \begin{pmatrix} 1 & \rho_1^D & \rho_4^D \\ \rho_1^D & 1 & \rho_3^D \\ \rho_4^D & \rho_3^D & 1 \end{pmatrix}$$

and $\hat{\omega} = (\hat{\omega}_o, \hat{\omega}_1, \hat{\omega}_2)$ is given by

$$\hat{\omega} = R^{-1} \rho^D(B) \begin{pmatrix} Z_T \\ Z_{T+1} \\ Z_{T+4} \end{pmatrix} .$$

Dropping, for notational simplicity, the superscript "D" from the dual autocorrelations, the estimator ω_o is found to be

$$\hat{\omega}_o = |R|^{-1}\{(1-\rho_3^2)\rho(B)Z_T-(\rho_1-\rho_3\rho_4)\rho(B)Z_{T+1}+(\rho_1\rho_3-\rho_4)\rho(B)Z_{T+4}\} , \quad (8.2)$$

where

$$|R| = 1+2\rho_1\rho_3\rho_4-\rho_1^2-\rho_3^2-\rho_4^2 .$$

Since the coefficient of Z_T in $\rho(B)Z_T$, $\rho(B)Z_{T+1}$, and $\rho(B)Z_{T+4}$ is, respectively, 1, ρ_1 and ρ_4, it is easily seen that the coefficient of Z_T in (8.2) is 1. Similarly, the coefficients of Z_{T+1} and Z_{T+4} in (8.2) are seen to be zero, so that the estimator of z_T

$$\hat{z}_t = Z_T - \hat{\omega}_o$$

does not depend on the three arbitrary numbers Z_T, Z_{T+1} and Z_{T+4}.

REFERENCES

Abraham, B. and Box, G.E.P. (1979), "Bayesian analysis of some outliers problems in time series," Biometrika, 66, 2, 229–36.

Box, G.E.P., Hillmer, S.C. and Tiao, G.C. (1978), "Analysis and modeling of seasonal time series" in Seasonal Analysis of Economic Time Series, ed. A. Zellner, Washington, D.C.: U.S. Dept. of Commerce–Bureau of the Census, 309–334.

Box, G.E.P. and Jenkins, G. (1976), Time Series Analysis. Holden–day.

Box, G.E.P. and Tiao, G.C.(1975),"Intervention analysis with applications to economic and environmental problems," Journal of the American Statistical Association, 70, 70–79.

Cleveland, W.P. (1972), "The inverse autocorrelations of a time series and their applications," Technometrics 14, 277–98.

Cleveland, W.P. and Tiao, G.C. (1976), "Decomposition of seasonal time series: A model for the X–11 program," Journal of the American Statistical Association, 71, 581–587.

Harvey, A.C. (1981), Time Series Models, Philip Allan.

Jones, R.H. (1980), "Maximum likelihood fitting of ARMA models to time series with missing observations," Technometrics 22, 389–95.

Kohn, R. and Ansley, C.F. (1983), "Fixed interval estimation in state space models when some of the data are missing or aggregated," Biometrika, 70, 3, 683–8.

Little, R.J.A.. and D.B. Rubin (1987), Statistical Analysis with Missing Data, New York: John Wiley & Sons.

Maravall, A. (1987), "On minimum mean squared error estimation of the noise in unobserved component models," Journal of Business and Economic Statistics, 5, 115–120.

Miller, R.B. and O.M. Ferreiro (1984), "A strategy to complete a time series with missing observations," in Time Series Analysis of Irregularly Observed Data, Ed. E. Parzen, New York: Springer–Verlag.

Peña, D. (1984), "Influential observations in time series," Mathematics Research Center Technical Report, 2718. University of Wisconsin–Madison.

Peña, D. (1987), "Measuring the importance of outliers in ARIMA models," New Perspectives in Theoretical and Applied Statistics, M. Puri et al (editors), New York: J. Wiley.

DOCUMENTOS DE TRABAJO (1):

8501 **Agustín Maravall:** Predicción con modelos de series temporales.
8502 **Agustín Maravall:** On structural time series models and the characterization of components.
8503 **Ignacio Mauleón:** Predicción multivariante de los tipos interbancarios.
8504 **José Viñals:** El déficit público y sus efectos macroeconómicos: algunas reconsideraciones.
8505 **José Luis Malo de Molina y Eloísa Ortega:** Estructuras de ponderación y de precios relativos entre los deflactores de la Contabilidad Nacional.
8506 **José Viñals:** Gasto público, estructura impositiva y actividad macroeconómica en una economía abierta.
8507 **Ignacio Mauleón:** Una función de exportaciones para la economía española.
8508 **J. J. Dolado, J. L. Malo de Molina y A. Zabalza:** Spanish industrial unemployment: some explanatory factors *(versión inglés)*. El desempleo en el sector industrial español: algunos factores explicativos *(versión español)*.
8509 **Ignacio Mauleón:** Stability testing in regression models.
8510 **Ascensión Molina y Ricardo Sanz:** Un indicador mensual del consumo de energía eléctrica para usos industriales, 1976-1984.
8511 **J. J. Dolado y J. L. Malo de Molina:** An expectational model of labour demand in Spanish industry.
8512 **J. Albarracín y A. Yago:** Agregación de la Encuesta Industrial en los 15 sectores de la Contabilidad Nacional de 1970.
8513 **Juan J. Dolado, José Luis Malo de Molina y Eloísa Ortega:** Respuestas en el deflactor del valor añadido en la industria ante variaciones en los costes laborales unitarios.
8514 **Ricardo Sanz:** Trimestralización del PIB por ramas de actividad, 1964-1984.
8515 **Ignacio Mauleón:** La inversión en bienes de equipo: determinantes y estabilidad.
8516 **A. Espasa y R. Galián:** Parsimony and omitted factors: The airline model and the census X-11 assumptions *(versión inglés)*. Parquedad en la parametrización y omisiones de factores: el modelo de las líneas aéreas y las hipótesis del census X-11 *(versión español)*.
8517 **Ignacio Mauleón:** A stability test for simultaneous equation models.
8518 **José Viñals:** ¿Aumenta la apertura financiera exterior las fluctuaciones del tipo de cambio? *(versión español)*. Does financial openness increase exchange rate fluctuations? *(versión inglés)*.
8519 **José Viñals:** Deuda exterior y objetivos de balanza de pagos en España: Un análisis de largo plazo.
8520 **José Marín Arcas:** Algunos índices de progresividad de la imposición estatal sobre la renta en España y otros países de la OCDE.
8520 **José Marín Arcas:** Algunos índices de progresividad de la imposición estatal sobre la renta en España y otros países de la OCDE.
8601 **Agustín Maravall:** Revisions in ARIMA signal extraction.
8602 **Agustín Maravall y David A. Pierce:** A prototypical seasonal adjustment model.
8603 **Agustín Maravall:** On minimum mean squared error estimation of the noise in unobserved component models.
8604 **Ignacio Mauleón:** Testing the rational expectations model.
8605 **Ricardo Sanz:** Efectos de variaciones en los precios energéticos sobre los precios sectoriales y de la demanda final de nuestra economía.
8607 **José Viñals:** La política fiscal y la restricción exterior. (Publicada una edición en inglés con el mismo número).
8608 **José Viñals y John Cuddington:** Fiscal policy and the current account: what do capital controls do?
8609 **Gonzalo Gil:** Política agrícola de la Comunidad Económica Europea y montantes compensatorios monetarios.
8610 **José Viñals:** ¿Hacia una menor flexibilidad de los tipos de cambio en el sistema monetario internacional?
8701 **Agustín Maravall:** The use of ARIMA models in unobserved components estimation: an application to spanish monetary control.

8702 **Agustín Maravall:** Descomposición de series temporales: especificación, estimación e inferencia (Con una aplicación a la oferta monetaria en España).
8703 **José Viñals y Lorenzo Domingo:** La peseta y el sistema monetario europeo: un modelo de tipo de cambio peseta-marco.
8704 **Gonzalo Gil:** The functions of the Bank of Spain.
8705 **Agustín Maravall:** Descomposición de series temporales, con una aplicación a la oferta monetaria en España: Comentarios y contestación.
8706 **P. L'Hotellerie y J. Viñals:** Tendencias del comercio exterior español. Apéndice estadístico.
8707 **Anindya Banerjee y Juan Dolado:** Tests of the Life Cycle-Permanent Income Hypothesis in the Presence of Random Walks: Asymptotic Theory and Small-Sample Interpretations.
8708 **Juan J. Dolado y Tim Jenkinson:** Cointegration: A survey of recent developments.
8709 **Ignacio Mauleón:** La demanda de dinero reconsiderada.
8801 **Agustín Maravall:** Two papers on arima signal extraction.
8802 **Juan José Camio y José Rodríguez de Pablo:** El consumo de alimentos no elaborados en España: Análisis de la información de Mercasa.
8803 **Agustín Maravall y Daniel Peña:** Missing observations in time series and the «dual» autocorrelation function.

(1) Los Documentos de Trabajo anteriores a 1985 figuran en el catálogo de publicaciones del Banco de España.

Información: Banco de España
Publicaciones. Negociado de Distribución y Gestión
Teléfono 446 90 55, ext. 2180
Alcalá, 50. 28014 Madrid