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AN APPLICATION OF NONLINEAR TIME SERIES FORECASTING

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ABSTRACT

By means of a real application, it is seen how ARIMA forecasts can be improved when non–linearities are present. The ACF of the squared residuals provides a convenient tool to check the linearity assumption. Then it is seen how simple bilinear models capture some of the non–linearity. The detection of non–linearity and the forecast improvement appear to be rather robust with respect to changes in the linear and bilinear specification. Finally, what bilinear models seem to capture are periods of atypical behavior or sequences of outliers.

1. INTRODUCTION

Forecasting using the Box-Jenkins methodology has become a well-established universal practice. The process of selecting an ARIMA model includes diagnostic checks which mostly rely on the autocorrelation function (ACF) of the fitted residuals. Although it is well known that, if the process is non-linear, lack of autocorrelation does not imply independence, a check of the linearity assumption is hardly ever performed(1).

Granger (1981) discusses situations in which what appears to be white-noise under covariance analysis can still be somewhat forecasted. A particular case he considers is related to bilinear processes. These were recently introduced in applied time series by Granger-Andersen (1978). They have received of late some attention(2), although they still appear to be far from practical use, with few reported applications. Bilinear time series models appear to have some theoretical foundation as approximation of more general non-linear models, and in some sense "can be regarded as a natural non-linear extension of the ARMA models" (Priestley,1981). For many of them, the ACF of the variable looks like that of white-noise, yet the series can be forecasted using its own past. Thus if the ARIMA residuals were to present such a bilinear structure, it would be possible to improve upon ARIMA forecasts.

Although the point is clear, the question still remains: is it likely to be of practical interest? This translates fundamentally into two requirements: first, the detection of non-linearity in a relatively easy way, and second, the ability to capture some of that non-linearity with simple (parsimonious) bilinear models.

In this paper, using an actual forecasting application, we try to address both issues. We conclude that

an additional tool can be easily incorporated to the diagnostic of ARIMA models, as a check on the linearity assumption. This check is trivial to compute and appears to behave rather properly. Then we shall see how simple bilinear models are able to capture some of the non-linearity. Furthermore, the detection of non-linearity and the forecast improvement achieved through the bilinear model appear to be rather robust with respect to (relatively small) changes in the linear and bilinear specification. Finally, we observe that what bilinear models seem to capture are periods of atypical behavior (or sequences of outliers), which affect the series occasionally and are not accounted for by the ARIMA model[3].

2. THE APPLICATION

The application we shall discuss is embodied within the Bank of Spain monetary control set-up.

Short-term monetary control is based on monthly targets for the rate of growth of the money supply. The money supply has two components: currency and an aggregate deposit component. Currency is treated mostly as exogenous. Control is, therefore, based on the relationship between the instruments used by the Bank and the deposit component, which plays the role of an intermediate target. The move from the target variable (money supply) to the intermediate one simply consists of substracting the currency forecast. Since currency demand is passively accomodated, at the Bank of Spain this forecast is obtained through ARIMA models for series with a ten-day observation period. This period corresponds to the bank data reporting frequency and allows intramonth information to be used in updating monthly forecasts[4]. We shall see if those currency forecasts can be improved by using bilinear models.

Insofar as the world is non-linear and linearity is a first-order approximation, since bilinear models may represent a second-order approximation, we might expect some improvement. But then, could this be achieved with parsimonious, easy-to-handle, ones?

3. ARIMA ESTIMATION

The 10-day currency series is an average of the daily series, obtained from bank statements, reported three times a month. The series is displayed in Table 1 and the last 150 values are shown in figure 1. For identification and estimation purposes, we consider the seven-year period 1974-80[5]. The forecasting exercise will cover the first semester of 1981. If y_t denotes the 10-day series, stationarity seems to be achieved through the transformation:

$$z_t = \nabla\nabla_{36} \log y_t \quad ,$$

roughly, the annual difference in the 10-day rate of growth. The autocorrelation function (ACF) of z_t is displayed in figure 2. The chosen ARIMA model was:

$$z_t = (1-\theta_1 B-\theta_9 B^9 -\theta_{18}B^{18}-\theta_{27}B^{27}-\theta_{36}B^{36}) \; a_t \; , \qquad (1)$$

with $\underline{\hat{\theta}}= (0.98, -.233, -.205, -.296, .237)$,

$\underline{t} = (1.63, -3.44, -2.98, -4.14, 3.23)$[6] ,

and residual variance $.561(10^{-4})$. The ACF of the residuals is shown in figure 3 and the last 150 values of a_t are plotted in figure 4. The forecastability measure $R^2 = 1 - \frac{\sigma_a^2}{\sigma_a^2}$ is equal to .18, and the standard deviation of the 1-step-ahead forecast error represents, roughly, .74 percent of the level of the series y_t.

4. CHECKING THE LINEARITY ASSUMPTION

The ACF of a_t looks like that of white-noise, but what if the series is nonlinear and the residuals simply uncorrelated but not independent?

The skewness of a_t is equal to .008 and kurtosis equals 3.545. Since the asymptotic standard deviations of these estimates are .167 and .334, respectively, the distribution of a_t looks rather symmetric, maybe slightly leptokurtic.

What we would like to have is a way of checking the linearity hypothesis, easy to compute, that does not require a specific alternative model. Granger suggests looking at the ACF of a_t^2. If a_t is independent, so will be a_t^2. But if a_t is not independent (and the model is nonlinear), this is likely to show in the ACF of a_t^2, which, in general, will not be that of white-noise.

Since it is trivial to compute, I did so(7). Figure 5 displays the ACF of a_t^2. There is some increase for seasonal and possibly low-order lags. In fact, to avoid effects due to "linear misspecification", I used four different ARIMA models, in three different computer packages, using both the conditional LS and backcasting options. For all of them, ρ_1 lay in the interval $[.07,.16]$, ρ_2 in $[.13,.18]$, ρ_{35} in $[.13,.15]$, ρ_{36} in $[.16,.20]$ and all the other ρ_k's were small(8). Thus minor changes in the ARMA estimation had little effect on the test. There is an underlying reason for this, which also makes the test more interesting.

<u>Lemma</u>: Let z_t be a linear (Gaussian) stationary process, then

$$\rho_k(z_t^2) = \rho_k(z_t)^2 \quad , \quad k = 0, \pm 1, \ldots \quad .$$

Proof: The moment generating function for z_t and z_{t-k} is given by:

$$m(t_1, t_2) = \exp. \{\tfrac{1}{2} \sigma_z^2 (t_1^2 + t_2^2 + 2\rho_k t_1 t_2)\} .$$

Since

$$E\, z_t^2\, z_{t-k}^2 = \left. \frac{\delta^4 m}{\delta t_1^2 \quad \delta t_2^2} \right|_{\underline{t} = \underline{0}} = \sigma_z^4 (1 + 2\rho_k^2)$$

and $\mathrm{Var}(z_t^2) = 2\sigma_z^4$, substituting for both expressions in

$$\rho_k(z_t^2) = \frac{E\, z_t^2\, z_{t-k}^2 - \sigma_z^4}{\mathrm{Var}(z_t^2)} ,$$

the result in the lemma is obtained.

Thus, for example, if we square an ARMA variable, we are also squaring its autocorrelations. If the process is linear, but wrongly specified, the misspecified residuals will also be linear, likely with relatively small autocorrelations. When squared, these will become negligible.

Comparing the ACF of a_t^2 with that of a_t, there is some evidence of nonlinearity for low-order and seasonal lags, and the rest of the autocorrelations seem rather small (not including ρ_1, ρ_2, ρ_{35} and ρ_{36}, the Q-statistics for the first forty autocorrelations were $Q(a_t) = 27.3$ and $Q(a_t^2) = 18.8$).

Before proceeding further in the application, a few general remarks are appropriate:

A) The lemma applies to any linear process, hence the test could conceivably be carried out directly on z_t. Since it implies that, for a linear stationary process,

$$|\rho_k(z_t)| > \rho_k(z_t^2) > 0 \quad , \quad \forall k \neq 0 \quad ,$$

unless both are zero, an increase in some autocorrelations when the variable is squared would imply non-linearity. However, there is a reason that makes it preferable to look at the ACF of the linearly pre-whitened series. An example will illustrate the problem. Consider the model:

$$z_t = a_t - \theta a_{t-1} \quad (2)$$

$$a_t = \beta a_{t-2}\, \varepsilon_{t-1} + \varepsilon_t \quad (3)$$

with $\varepsilon_t \sim \text{Niid}\ (0,\sigma_\varepsilon^2)$. Equation (3) represents a bilinear process. It is seen that β is not unit free, its dimension being the inverse of that of z_t. For analytical discussion a convenient standarization is $\sigma_\varepsilon^2 = 1$, in which case (3) is stationary when $|\beta| < 1$. If estimates of past ε_t's are available, onestep-ahead forecasts of a_t can be obtained through

$$\hat{a}_t(1) = \beta a_{t-1}\, \varepsilon_t \quad .$$

However, it is easily seen that a_t is uncorrelated at all lags, so that its ACF is that of white-noise. On the contrary, as we shall see later, the ACF of a_t^2 consists of two alternating exponentially decreasing functions. Within the region $|\beta| < .76$ (for which the second moments of a_t^2 exist)

$$\rho_2(a_t^2) > \rho_k(a_t^2) \quad , \quad k \neq 2 \quad ,$$

always holds, except when both are zero. Non-linearity would be detected mainly by a relatively large value of $\rho_2(a_t^2)$. For the z_t variable, it can be seen that:

$$\rho_2(z_t) = 0$$

$$\rho_2(z_t^2) = \beta^2 \left[1 - \frac{\theta^2(1-3\beta^4)(1+5\beta^2)}{1+\theta^4+\theta^2(1-3\beta^4)\ \ 2(1+3\beta^2)} \right] \quad ,$$

so that, for any (non-zero) value of θ, $\rho_2(a_t^2) = \beta^2 > \rho_2(z_t^2)$. Figure 6 displays both autocorrelations as functions of β for θ=.8. Since, if linear, $\rho_2(z_t^2) = \rho_2(a_t^2) = 0$, the bilinear structure is more likely to be revealed using the a_t^2 series. If z_t^2 were to be used, lack of any increase in the ACF may be due to the complicated manner in which (3) and (4) interact, and not to the fact that the series is linear. However, when using a_t^2,this interaction disappears, since a_t is free from linear correlation In fact, there are general additional reasons for non-linearity tests to be performed preferably on the a_t series[9].

B) The lemma and previous discussion have dealt with theoretical ACF. In practice we use the standard estimates, such as in Box-Jenkins (1970). Is it likely that the underlying non-linearity can damage their precision in such a way as to invalidate linear estimation and detection of non-linearity?

For the model consisting of equation (3), together with

$$z_t = \phi z_{t-1} + a_t \quad ,$$

with ϕ = .5 and β = .4, 250 random samples of size 250 each, were drawn from a N(0,1) population. For this model, $\rho_1(a_t) = 0$, $\rho_1(z_t) = .5$ and $\rho_1(a_t^2) = .15$. The histograms for $\hat{\rho}_1(a_t)$, $\hat{\rho}_1(z_t)$ and $\hat{\rho}_1(a_t^2)$ are shown in figure 7, which also includes the asymptotic distribution of the first two under the linearity assumption. It is seen that non-linearity seems to have little effect on the distributions of $\hat{\rho}_1(z_t)$ and $\hat{\rho}_1(a_t)$. Also $\hat{\rho}_1(a_t^2)$ seems to be reasonably acceptable. The standard deviations for the three estimators were .068, .074 and .103, respectively, while $T^{-1/2}$=.064[10]. The estimator $\hat{\rho}_1(z_t^2)$ was considerably more erratic, with σ = .146.

C) Often, bilinear models tend to have relatively small values for $\rho(a_t^2)$. Thus, in order for $\hat{\rho}(a_t^2)$ to be able to detect non-linearity, the series should have a relatively large number of observations. In practice, for many economic time series, this may mean that the test based on the ACF of a_t^2 would be appropriate for series with relatively high frequency of observation. But then these are likely to be the series exhibiting more important non-linearities.

Two types of factors can be expected to operate: first, a statistical one, consisting of central limit-type effects, which render temporal aggregates more Normal (Anderson, 1971, section 7.7), and second, a geometric effect, which is simply the fact that quadratic approximations will perform better than linear ones. Therefore one would expect daily series to be more non-linear than 10-day ones, which in turn should be more non-linear than monthly ones. In our case, direct inspection of the daily series (figure 8) shows irreversibility, which is a clear indication of non-linearity[(11)].

Thus a comparison between the daily, 10-day and monthly series seemed appropriate to see what information is provided by the ACF of a_t^2.

Figure 9 displays the ACF of a_t and a_t^2 for the residuals of the monthly series. It is seen that the series a_t^2 appears to be white-noise, and the autocorrelations are of small size. Figure 10 compares the ACF of a_t and a_t^2 for the daily series[(12)]. There is a significant increase in the values of $\rho_k(a_t^2)$ for low order and seasonal lags.

Since the Q-statistics is an aggregate measure of a set of autocorrelations, and since the lemma implies that $Q_k(z_t^2) \leq Q_k(z_t)$ for any set of k autocorrelations, an increase in a Q value would be an indication of

non-linearity(13). The following table presents Q statistics for the residuals of the monthly series:

Monthly series	Q_{12}	Q_{24}	Q_{36}
a_t	6.3	16.6	20.5
a_t^2	9.5	18.9	20.6

They appear to behave rather linearly. For the daily series, a similar table yields:

Daily series	Q_{12}	Q_{313}	Q^*_{313}	Q_{626}
a_t	65.6	448.4	392.6	659.2
a_t^2	198.8	518.3	295.3	658.1

where Q_{12} is intended to capture the low-order autocorrelations and Q_{313} to incorporate the seasonal ones. The asterisk denotes the value of Q_{313} when the autocorrelations for k = 1, 2, 6, 77-79, 155-157 and 312-314 have been removed. There are noticeable increases in $Q_{12}(a_t^2)$ and $Q_{313}(a_t^2)$, which are due to increases in the autocorrelation for "sensible" lags (low-order and several seasonal ones, such as weekly, quarterly, semiannual and annual).

Looking at the results for the daily series, it appears that, for large k, linearity may imply smaller variances for the distribution of $\hat{\rho}_k(a_t^2)$. If such were to be the case, care should be taken when considering Q values which involve a large number of autocorrelations: the increase in a few $\hat{\rho}_k(a_t^2)$'s could cancel out with the decrease in the rest of them.

This seems to be evidenced in the behavior of Q for the daily series: the increase in Q_{12} is larger than the one in Q_{313}, despite the fact the non-linearity shows up at k = 78, 156 and 313; moreover, both Q_{626} are practically identical.

For the 10-day series the following Q-values were obtained:

10-day series	Q_{12}	Q_{36}
a_t	8.3	31.5
a_t^2	13.6	43.0

To summarize, the ACF of a_t^2 indicates that the monthly series behaves rather linearly while the daily one is clearly non-linear, with the 10-day series standing in between. Thus the ACF of a_t^2 behaves quite properly. All in all, considering its computational simplicity, its robustness with respect to linear specification and the information it may provide, the ACF of a_t^2 seems to offer a reasonable tool to add to the standard diagnostic check of ARIMA fits, to check for the validity of the linearity assumption.

5. IDENTIFICACION

Back to our application, we shall center on the 10-day series. Is it possible to capture the (relatively small) detected non-linearity with parsimonious bilinear models?

Two general approaches are possible: first, a direct one, in which a general bilinear model is fitted to z_t, and second, a two-stage ("forecasting white-noise") approach, in which the linear innovations are first obtained and then a bilinear model is fit to them. Although less general, we followed the second approach. It is at present computationally easier and it has a nice feature: Let

$$z_t = \psi(B)\ a_t \tag{4}$$

$$a_t = \varepsilon_t + N(\varepsilon_{t-k}\ x_{t-j})\ ,\ k,j>0\ , \tag{5}$$

where a_t is uncorrelated, ε_t white-noise and N denotes a bilinear term. Roughly, we can write:

$$z_t = l_t + n_t \quad ,$$

where $l_t = \psi(B)\ \varepsilon_t$ is a linear term and n_t a non-linear expression. Then it is easily seen that:

$$\rho_k(z_t) = \rho_k(l_t) \quad , \quad \forall\ k \quad .$$

Thus, in terms of the first stage, we do not have to worry about non-linearity since the ACF of z_t identifies correctly $\psi(B)$.

From a general point of view, we are dealing with a linear function of non-Gaussian variables. Having obtained the linear function (4), we are interested now in finding a bilinear process that (a) is uncorrelated and (b) can generate ACFs similar to the one obtained for a_t^2. In this sense, the ACF of a_t^2, besides detecting

non-linearities, provides an identification tool for the bilinear specification.

Consider the process given by (3), with $|\beta|<.76$ and $\sigma_\varepsilon^2 = 1$. Then it can be seen that

$$Ea_t = 0$$

$$\mathrm{Var}\ (a_t) = \frac{1}{1-\beta^2} \quad ,$$

$$\rho_k(a_t) = 0 \quad , \quad k \neq 0$$

so that, under covariance analysis a_t looks like white-noise. The ACF of a_t^2 is given by:

$$\rho_1(a_t^2) = \beta^2(1-3\beta^4) \quad ,$$

$$\rho_2(a_t^2) = \beta^2$$

$$\rho_k(a_t^2) = \beta^k \quad , \ k \text{ even} \quad ,$$

$$\rho_k(a_t^2) = \beta^{k-1}\ \rho_1(a_t^2) \quad , \quad k \quad \text{odd} \quad ,$$

which imply alternating exponentially decreasing functions, with initial conditions 1 and $\rho_1(a_t^2)$, and parameter equal to β^2. Notice that $\rho_2(a_t^2)>\rho_1(a_t^2)$. Figure 11 displays a typical pattern of the ACF of a_t^2. Since in our case this pattern characterizes both the low order and seasonal autocorrelations, this suggests the use of a model such as:

$$a_t = \beta_1\ a_{t-2}\ \varepsilon_{t-1} + \beta_2\ a_{t-36}\ \varepsilon_{t-35} + \varepsilon_t \quad , \qquad (6)$$

a rather parsimonious representation[14]. By generating series with (3), ACFs for a_t^2 with positive peaks at lags ρ_1, ρ_2, ρ_{35} and ρ_{36} were easily obtained. Furthermore, (3) with $\varepsilon_t \sim$ Niid implies a symmetric, slightly leptokurtic distribution for a_t. For example, for β = .5, the coefficient of kurtosis $\mu_4 = \dfrac{3(1-\beta^4)}{1-3\beta^4}$ is equal to 3.46.

6. ESTIMATION

In order to estimate (6), an appropriate standarization is to set $\sigma_a^2 = 1$. Maximum likelihhod estimators of β_1 and β_2 are obtained by minimizing (Priestley, 1981, p.881):

$$S(\underline{\beta}) = \sum_t \varepsilon_t^2 \quad .$$

In computing $S(\underline{\beta})$, starting values for the ε's were set equal to zero and subsequent values were computed recursively. In order to avoid the effect of the starting conditions, we used the last 150 values of the ε_t series in the computation of $S(\underline{\beta})$. Figure 12 plots the contours of $S(\underline{\beta})$ within the stationary region. The minimum is reached for $\hat{\beta}_1 = .02$, $\hat{\beta}_2 = -.22$ and the residual variance becomes .937[(15)]. In fact, if the complete ε_t series is used (not including the zero starting values), the estimators remain practically unchanged and $\sigma_\varepsilon^2 = .922$. On the other hand, if only the last 100 values are considered, $\hat{\beta}_2$ becomes to $-.28$ and $\sigma_\varepsilon^2 = .913$. Setting $\hat{\beta}_1 = 0$, $\hat{\beta}_2 = -.22$, in terms of the original series z_t, the forecastability measure R^2 increases by nearly 32%. Figures 13 and 14 display the ACF of ε_t and ε_t^2, respectively. The additional filter seems to have increased the covariance between neighboring values of the autocorrelation estimates. Also, the lag 2 autocorrelation is still present in the ε_t^2 series. A crude goodness-of-fit test is provided by comparing the statistics:

$$T \log\{\sigma_a^2/\sigma_\varepsilon^2\} = 13$$

to $\chi_2^2(.05) = 6$ (Priestley, 1981, p. 884). The significance of the model is due to the seasonal component. In fact, if a regression is run on (6), with ε_t replaced by the residuals from the bilinear fit, the t statistics for β_1 and β_2 are .23 and -2.87. Summarizing, estimation of (6) shows significant non-linearity at seasonal lags.

7. FORECASTING

For forecasting purposes we consider the first semester of 1981 (T = 18). The exercise will be performed under the standarization $\sigma_a = 1$.

If $\hat{z}^1_t(1)$ denotes the one-period ahead linear forecast obtained with (1), then it is easily seen that:

$$\text{a)} \quad \hat{z}_t(1) = \hat{z}^1_t(1) + \hat{a}_t(1) \quad , \qquad (7)$$

where $\hat{z}_t(1)$ is the final forecast, and $\hat{a}_t(1)$ is given by:

$$\hat{a}_t(1) = -.22\ a_{t-35}\ \varepsilon_{t-34} \quad , \qquad (8)$$

where we have set $\beta_1 = 0$. Also, it follows that

$$\text{b)} \quad \varepsilon_{t+1} = a_{t+1} - \hat{a}_t(1) = z_{t+1} - \hat{z}_t(1) \quad ,$$

so that ε_t is the one-step ahead prediction error of the currency series (in logs).

One-step ahead forecasts were obtained with the linear model (1) and then bilinear forecasts, given by (8), were added. The models were not re-estimated. However, adding the new 18 observations had a negligible effect on the ACF of z_t. The standard deviation of the ARIMA forecast error for the first semester of 1981 was equal to .82, hence for this period, ARIMA forecasts were particularly accurate. Figure 15 plots the forecast errors for both $\hat{z}^1_t(1)$ and $\hat{z}_t(1)$. The MSE decreases by close to 8%, and it is seen that most of the improvement is concentrated in the last two months.

For forecast horizons larger than one period, a result similar to (7) does not hold. For example, the two-period ahead forecast is given by:

$$\hat{z}_t(2) = \hat{z}^1_t(2) + \hat{a}_t(2) - \theta_1\ \hat{a}_t(1) \quad ,$$

so that the forecast error becomes:

$$e_t(2) = z_{t+2} - \hat{z}_t(2) = \varepsilon_{t+2} - \theta_1\ \varepsilon_{t+1} \ .$$

Hence:

$$\frac{V(e_t(2))}{V(e_t^1(2))} = \frac{\sigma_\varepsilon^2}{\sigma_a^2} \quad , \qquad (9)$$

where e_t^1 denotes the linear forecast error. Thus the relative improvement remains constant. Indeed, the results for the 2 and 3 steps ahead forecasts were virtually identical and we shall not discuss them. Notice that if $\beta_1 \neq 0$ this would not be true, since then

$$e_t(2) = \varepsilon_{t+2} + \beta_1 \, a_t \, \varepsilon_{t+1} - \theta_1 \, \varepsilon_t \quad ,$$

and therefore (9) would not hold.

8. CHANGES IN THE SPECIFICATION

1. Linear Specification

I mentioned before that several ARIMA estimations were tried. The largest differences were due to the use of "conditional least-squares" versus "backcasting" in the estimation phase. I redid the exercise with an ARIMA similar to (1), with $\theta_{36} = 0$, $\theta_1 = 0$, estimated through CLS. Although the residual variance is slightly greater ($\sigma_\varepsilon^2 = .596(10^{-4})$), the ACF of the residuals was cleaner, with smaller Q-values. The ACF of a_t^2 showed $\rho_1 = .16$, $\rho_2 = .16$, $\rho_{35} = .14$, $\rho_{36} = .20$ and all other autocorrelations were small. Thus non-linearity was slightly more noticeable[(16)]. Estimation of the bilinear model (6) yielded: $\beta_1 = .03$, $\beta_2 = -.16$, with $\sigma_\varepsilon^2 = .922$. Figure 16 plots the forecast errors for the first half of 1981. The improvement induced by the use of the bilinear model is similar to the one obtained before, though the decrease in MSE becomes now over 13%. Since the MSE of the ARIMA forecast was practically identical to the one obtained before, the final forecasts obtained in this case were more accurate. This suggests that if the series presents non-linearities, the linear estimation (i.e., the first step) should be performed preferably using the CLS option of ARIMA packages.

2. Bilinear Specification

But, besides misspecification of the linear model, there can also be misspecification of the bilinear one. Would (minor) changes in the latter affect the results much?

Besides model (6), I tried the following bilinear formulations:

$$a_t = \beta_1 a_{t-1} \varepsilon_{t-1} + \beta_2 a_{t-36} \varepsilon_{t-36} + \varepsilon_t \qquad (10\ a)$$

$$a_t = \beta_1 a_{t-2} \varepsilon_{t-2} + \beta_2 a_{t-36} \varepsilon_{t-36} + \varepsilon_t \qquad (10\ b)$$

$$a_t = \beta_1 a_{t-1} \varepsilon_{t-2} + \beta_2 a_{t-35} \varepsilon_{t-36} + \varepsilon_t \qquad (10\ c)$$

$$a_t = \beta_1 a_{t-2} \varepsilon_{t-1} + \beta_2 a_{t-36} \varepsilon_{t-36} + \varepsilon_t \qquad (10\ d)$$

$$a_t = \beta_1 a_{t-2} \varepsilon_{t-2} + \beta_2 a_{t-36} \varepsilon_{t-35} + \varepsilon_t \qquad (10\ e)$$

$$a_t = \beta_1 a_{t-2} \varepsilon_{t-1} + \beta_2 a_{t-36} \varepsilon_{t-1} + \varepsilon_t \qquad (10\ f)$$

Using Granger-Andersen terminology, (10 a and b) are diagonal models, (10 c) is subdiagonal, (10 d and e) are mixed ones and (10 f) is the only one that is completely uncorrelated[17]. All of them could generate ACFs for a_t^2 somewhat similar to our example. Except for (10 f), which did a bad job, the rest all improved some the linear results. Although differences were relatively minor, (a, b and e) produced smaller σ_ε^2 than (6), while (c and d) performed worse. For all of them, $\hat{\beta}_1$ was not significant. We summarize the results for (10 a).

Contours of $S(\underline{\beta})$ are shown in figure 17. Figure 18 exhibits the one-step ahead forecast errors when the model

$$a_t = -.2\, a_{t-36} \varepsilon_{t-36} + \varepsilon_t$$

is added to (1). Again, the improvement is concentrated over the same period, and the MSE of the forecasts is reduced by 9.2%. Roughly, the results seem rather robust with respect to changes in the bilinear specification.

The improvement achieved is relatively small, as one would expect from a second order type of approximation. But an 8% reduction in the MSE of the currency forecast is by no means irrelevant for monetary policy. Moreover, practical implementation of (8) is computationally trivial, and since $\hat{a}_{t+j}(1) = \hat{a}_t(j+1)$, $j = 0,1,\ldots,34$, it can be done at the beginning of a year, for the complete year ahead.

9. A FINAL COMMENT: OUTLIERS AND NONSTATIONARITY

If we look at the figures displaying the forecast errors, it is seen that the improvement is concentrated over the last two months. During this period forecasts were over-estimated due to an unexpected drop in currency. This drop continued during July and then recovered. What the bilinear model was able to capture, therefore, was part of a special type of effect that could not be accounted for by the simple linear model.

It appears, therefore, that bilinear models could be appropriate for series with sequences of outliers, where, on occasion, a different regime seems to apply. In fact, the way bilinear filters seem to operate is as follows: During the "normal regime", they are mostly inoperative. When atypical behavior sets in, they become operative, and perform some smoothing of outliers. Hence bilinear structures can be seen partly as filters for somewhat smoothing outliers.

This feature becomes more noticeable when considering nonstationary bilinear processes. Figure 19 displays series generated with

$$z_t = \beta z_{t-1}\, \varepsilon_{t-1} + \varepsilon_t \quad , \qquad (11)$$

for two random samples from $\varepsilon_t \sim Niid(0,1)$, for $\beta = 0$ and $\beta = 1$, the latter being a nonstationary value. Nonstationarity seems to be mostly associated with occasional blow-ups in variance, with an eventual return to a constant mean level[18]. This type of nonstationarity is rather different from the one associated with trends and with lack of convergence of the ACF. In fact, for many bilinear models, as they approach nonstationarity, the ACFs tend toward that of white-noise[19]. Hence bilinear models are able to produce series behavior which cannot be internally generated by linear models. They present therefore some features which could be of potential applied interest.

FOOTNOTES

(1) By linear model we shall denote a linear filter of Gaussian white-noise.

(2) See, for example, Priestley (1981), Subba Rao (1981) Pham-Tran (1981) and Gabr-Subba Rao (1981).

(3) A more complete version of the paper is contained in Maravall (1982). Some results are also contained in Maravall (1981).

(4) The period is also the one for which reserve requirements are set. The first two ten-day periods of a month cover the first twenty natural days. The third period is, therefore, a residual. For a more complete description of the control procedure, see Espasa and Pérez (1979).

(5) 1974 is the first year for which a complete set of homogenous data is available.

(6) Although θ_1 was not significantly different from zero, it was kept because it improved slightly the forecasts for 1981.

(7) More sophisticated test, such as the ones based on the polyspectra (Subba Rao - Gabr, 1980), could also be applied. The nice feature of the ACF of a_t^2 is that it is inmediately available to any ARIMA user.

(8) The largest difference was caused by the use of CLS, versus backcasting, with the former producing larger values for $\rho_1(a_t^2)$. The model we use in the discussion is (1) estimated in the Speakeasy routines, with backcast option. In fact the residuals for this model were the ones that displayed less evidence of non-linearity.

(9) See, for example, Granger (1979) and Davies-Spedding-Watson (1980).

(10) The three sample estimates present a downward bias, analogously to the case of linear processes (see Kendall, 1973, chap. 7)

(11) The series jumps to a maximum and then slowly declines (Cox, 1981).

(12) The daily series covers the shorter period 1977-80 and does not include Sundays.

(13) Since we are only interested in a measure of aggregate value, we use the original Box-Pierce Q-statistics. In fact, since, under the linearity assumption, $\hat{\rho}(a_t^2) \,\dot{\sim}\, N(0, \frac{1}{T})$, the Q statistics for the a_t^2 series is likely to be asymptotically distributed also as a χ^2 variable.

(14) Strictly speaking, a_t defined by (4) is not uncorrelated at all lags. There is a nonzero ρ_{34}, but in our application we need not worry about it since it will have a value of .004.

(15) Notice that, in terms of the standarization $\sigma_\varepsilon^2=1$, the estimates would be slightly larger.

(16) This is possibly related to the fact that backcasting is a linear operation which would induce linearity at the beginning of the series.

(17) Though non-zero autocorrelations were in all cases negligible.

(18) Alternative time series models that appear to present somewhat similar second-order behavior have been developed by Engle and Kraft (1981).

(19) Model (11) presents this feature. Its only non-zero autocorrelation is $\rho_1(z_t)$, which reaches a maximum for $\beta \simeq .6$ and tends to zero as $\beta \rightarrow 1$.

Table 1

Ten-day Currency Series

(1974-80 plus first semester 1981)

364461	341413	328620	344680	334563	331340	356045	343981
339566	357619	341962	341025	363956	352310	346394	369615
358344	357835	391569	400517	401153	412532	395092	379739
398274	381734	374367	395066	381069	368151	392456	374457
366658	397381	397412	434195	424712	399718	382700	406182
388360	384440	414632	402104	390214	416302	407888	397808
428724	414536	409943	437785	422819	424082	461271	475294
478949	493082	470235	453391	475167	454763	444053	470502
453034	440360	472313	450855	446905	478236	475995	511699
501217	468852	451873	478621	459036	455708	487215	475383
464047	500677	483000	470400	503107	482352	476104	507234
493310	490404	535447	550944	553465	576486	546829	524090
553334	532592	514461	551685	529344	511974	542964	517454
503822	550867	546553	592828	591549	552213	532208	565463
543178	534563	574911	559083	545531	592300	571900	556313
596642	573676	562711	614338	601357	588116	652324	669183
688481	715611	681689	650405	690881	665255	642875	689703
660064	642431	680471	655918	640290	700145	690916	747349
752041	708151	681175	723834	697778	682251	736598	719949
698522	760024	735034	719198	772313	746134	728223	784439
762081	754148	840779	847971	857405	886908	853674	811078
862790	832875	805340	858323	826514	797056	847479	817151
789161	868966	857722	915211	915084	864998	824811	880715
852467	831918	905809	883963	856501	925085	895098	867755
917201	892533	859498	923734	901297	882113	971241	967730
964281	1005793	974540	926331	972033	946000	913321	966259
940035	899678	952559	922335	887411	970700	962685	1010413
1005845	961846	911807	968041	941214	906874	976651	955881
923593	993338	978027	943232	1015262	994002	966669	1046085
1025264	1010727	1110303	1110236	1103570	1162776	1128276	1077459
1128056	1097629	1054622	1115968	1084900	1037600	1098000	1063400
1026094	1115121	1106440	1151435	1155568	1105800	1052278	1111924
1079268	1054281	1128637	1105145	1068464	1147057	1126353	1089787
1160336	1133221	1096754	1174281	1151458	1123718		

Figure 1

Ten-days Currency Series (y_t)

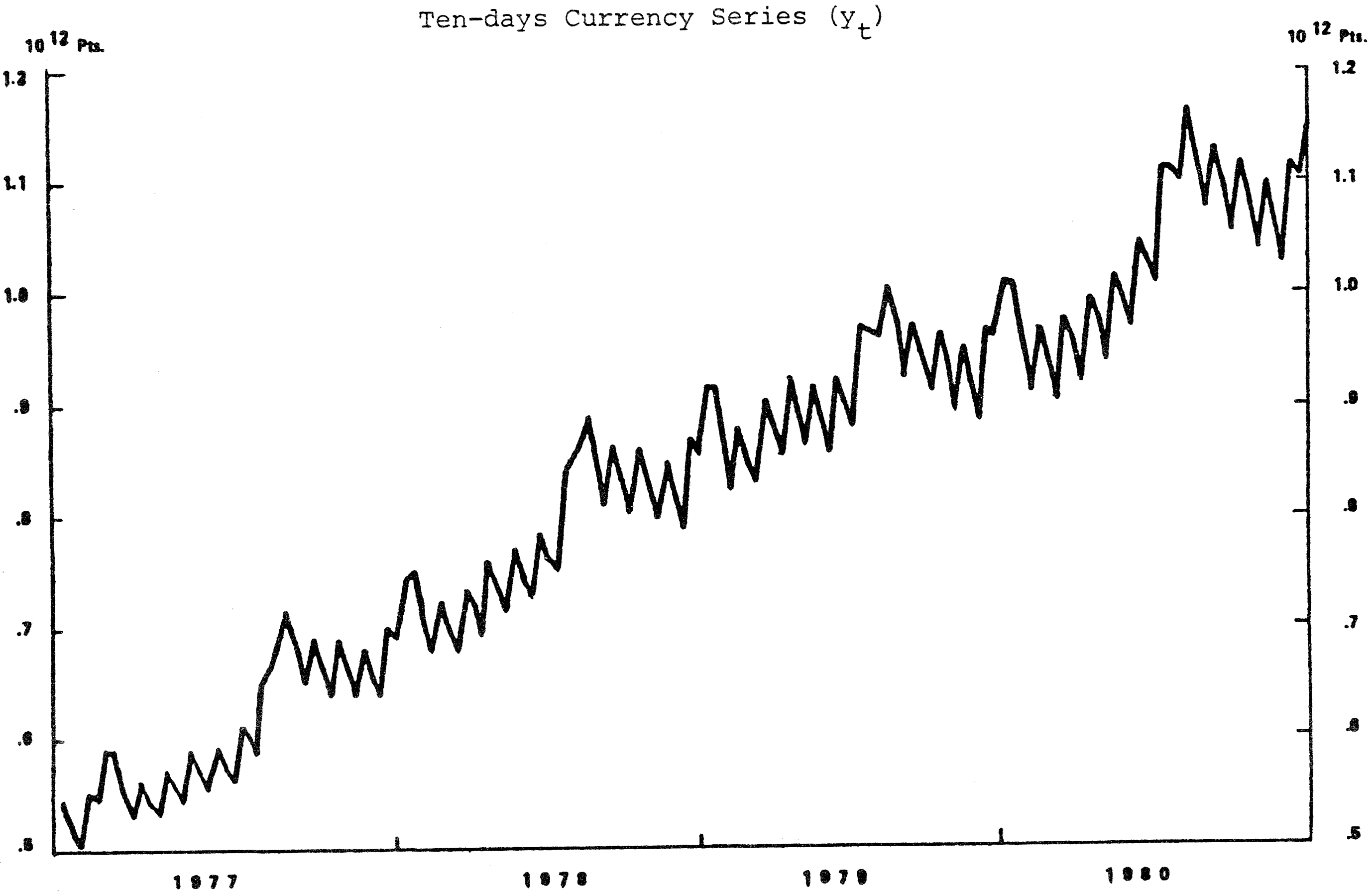


Figure 2

ACF of $z_t = \nabla\nabla_{36} \log y_t$

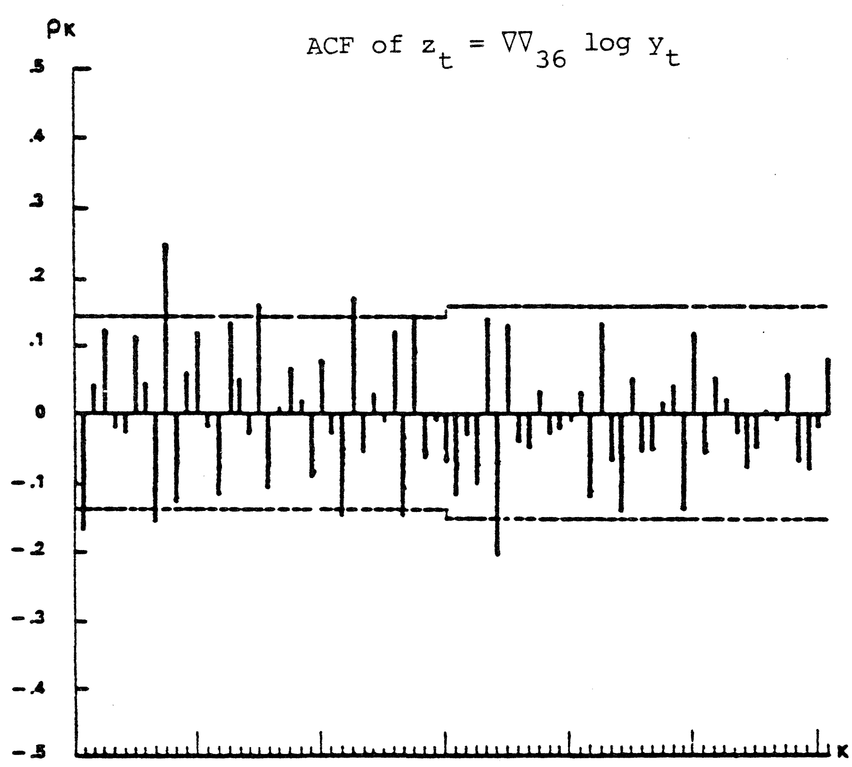


Figure 3

ACF of ARIMA Residuals (a_t)

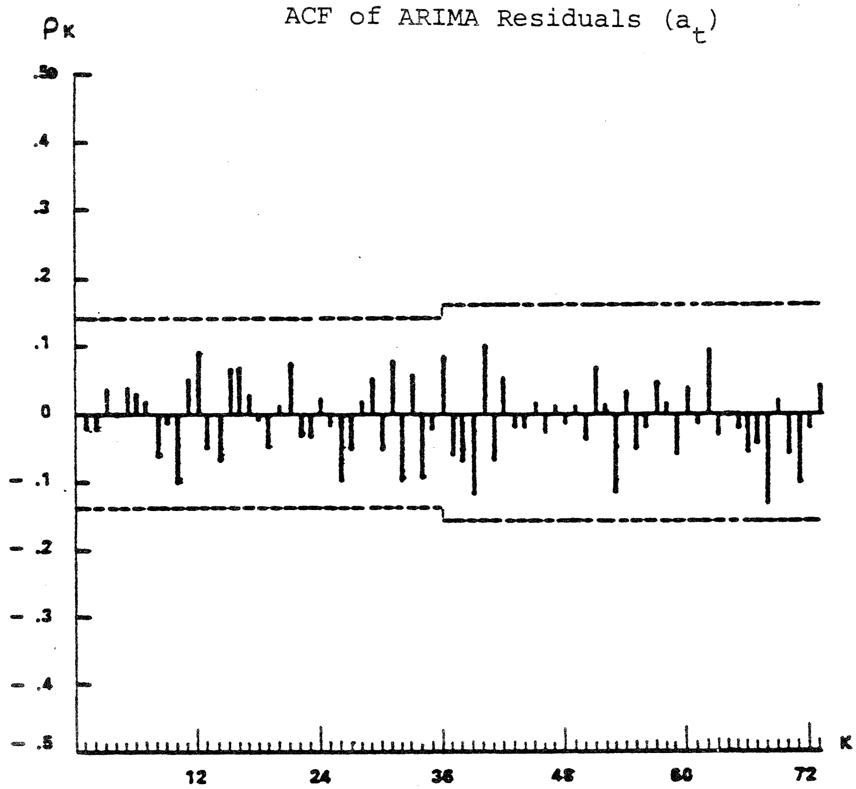


Figure 4

ARIMA Residuals (a_t)

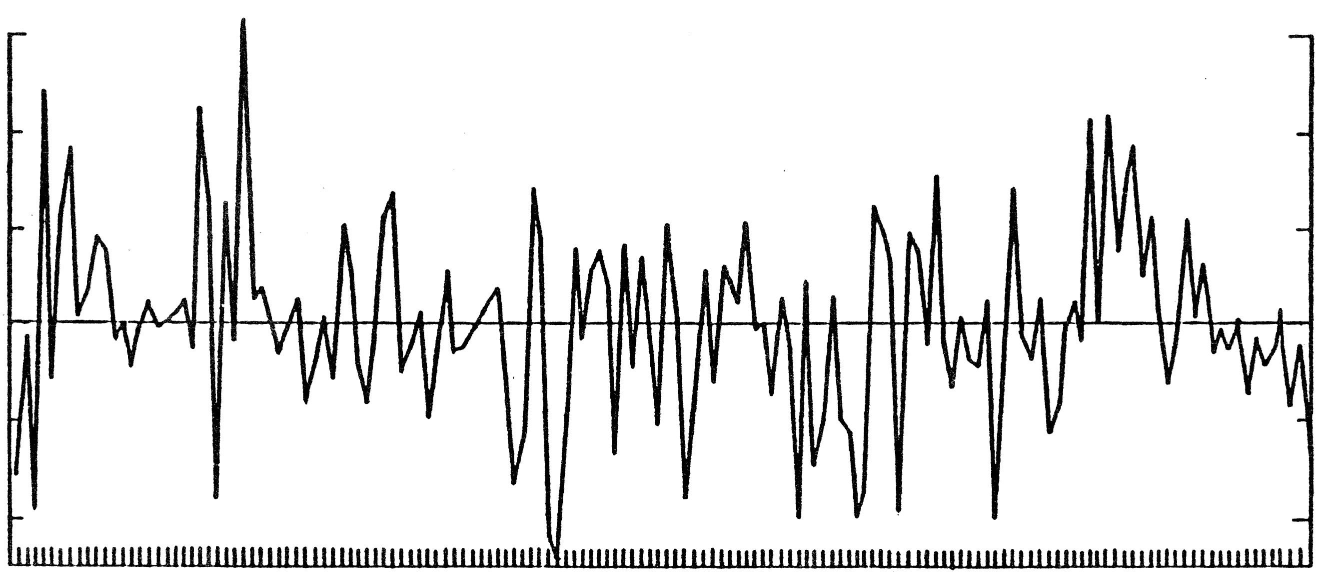


Figure 5

ACF of (a_t^2)

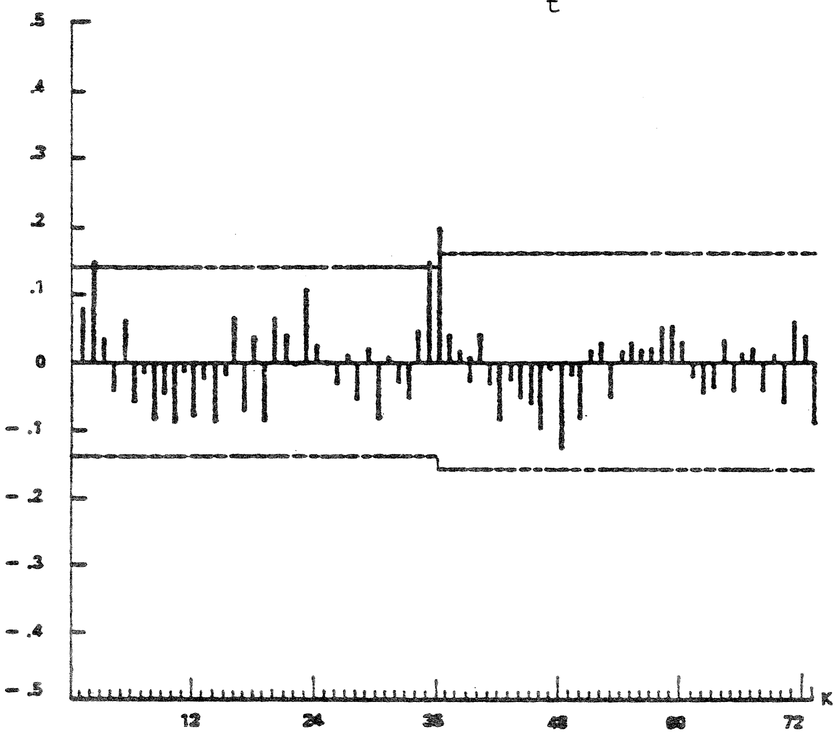


Figure 6

Lag two Autocorrelation for (a_t^2) and (z_t^2)

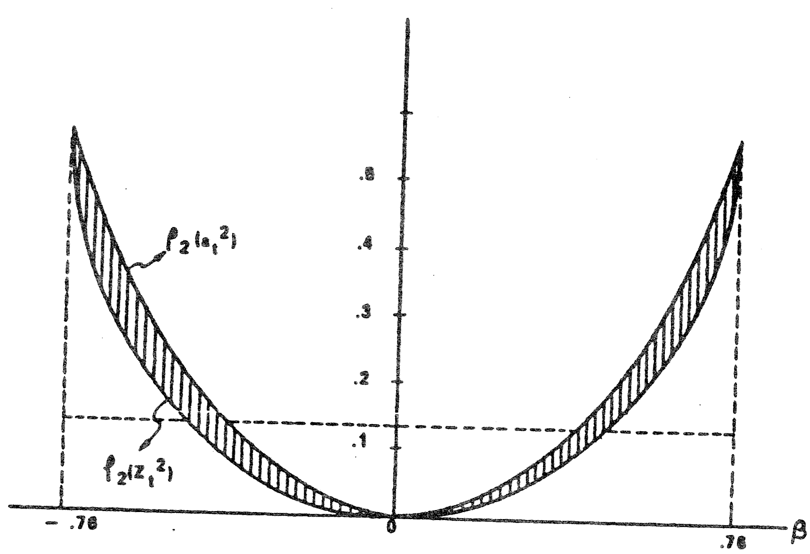


Figure 7

Histograms

a) $\hat{\rho}_1(z_t)$ b) $\hat{\rho}_1(a_t)$ c) $\hat{\rho}_1(a_t^2)$

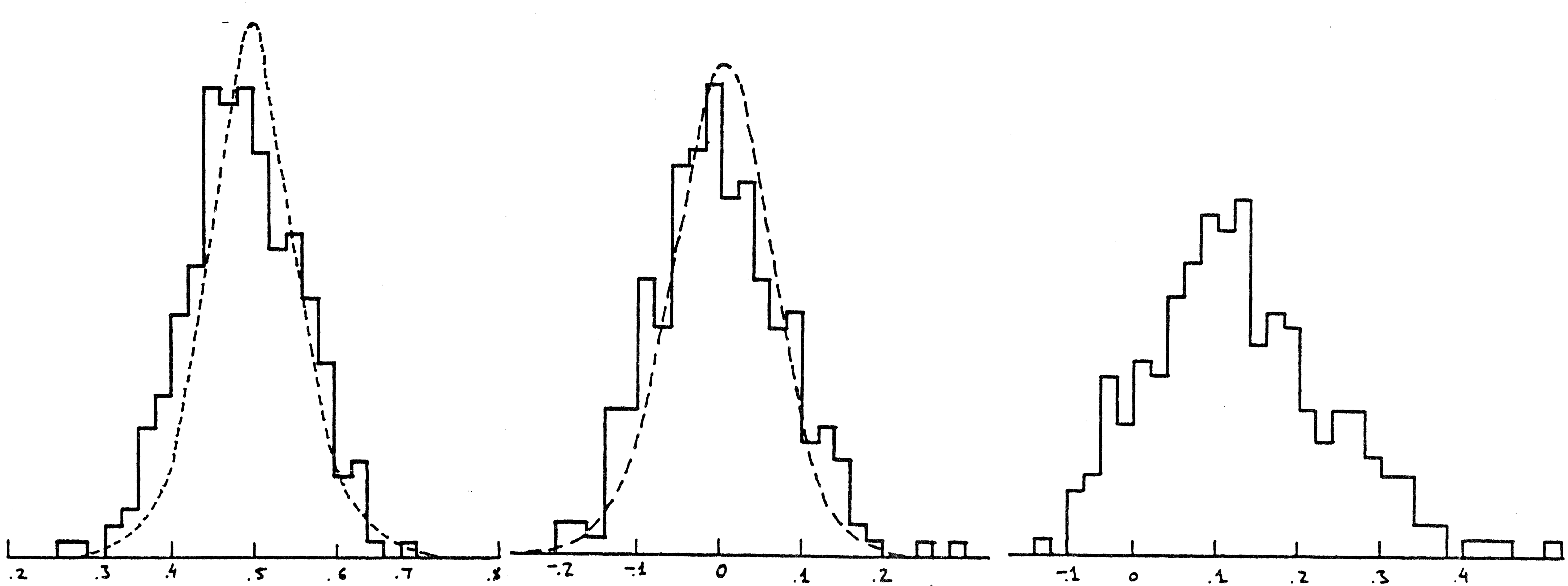


Figure 8

Daily Currency Series

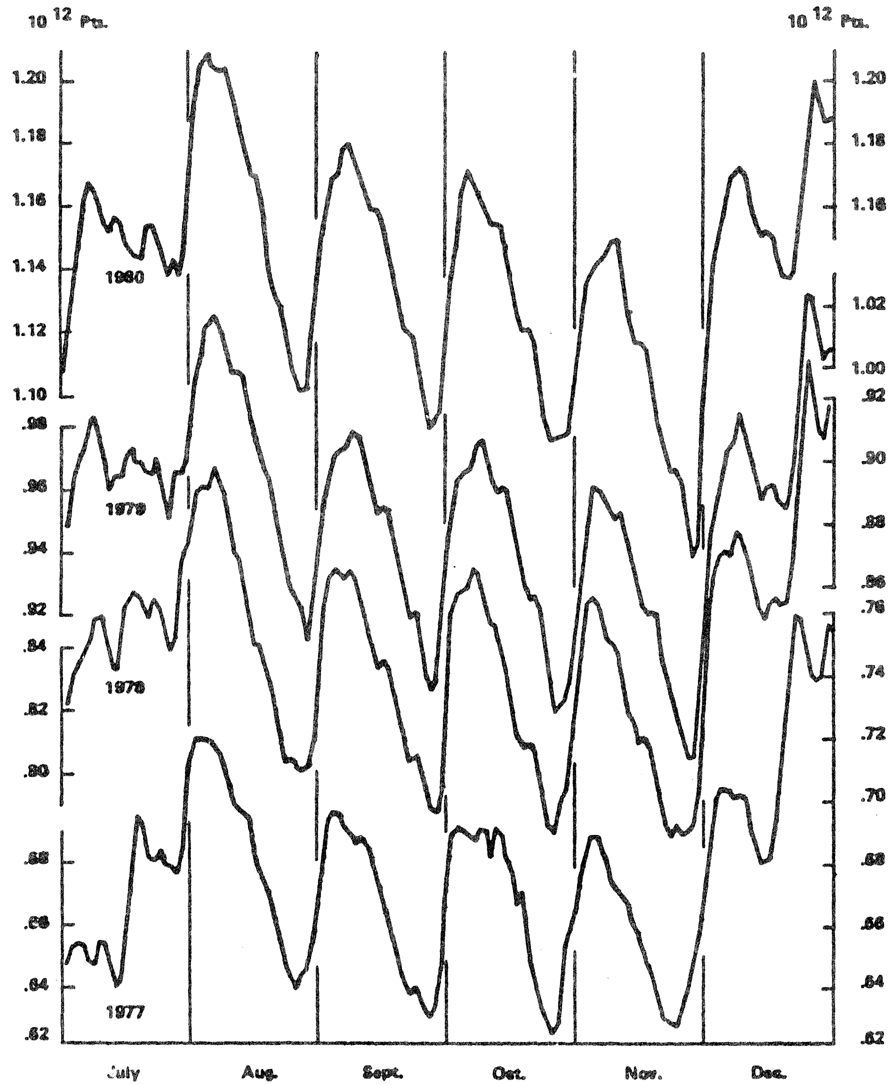


Figure 9

ACF of (a_t) and (a_t^2): Monthly Series

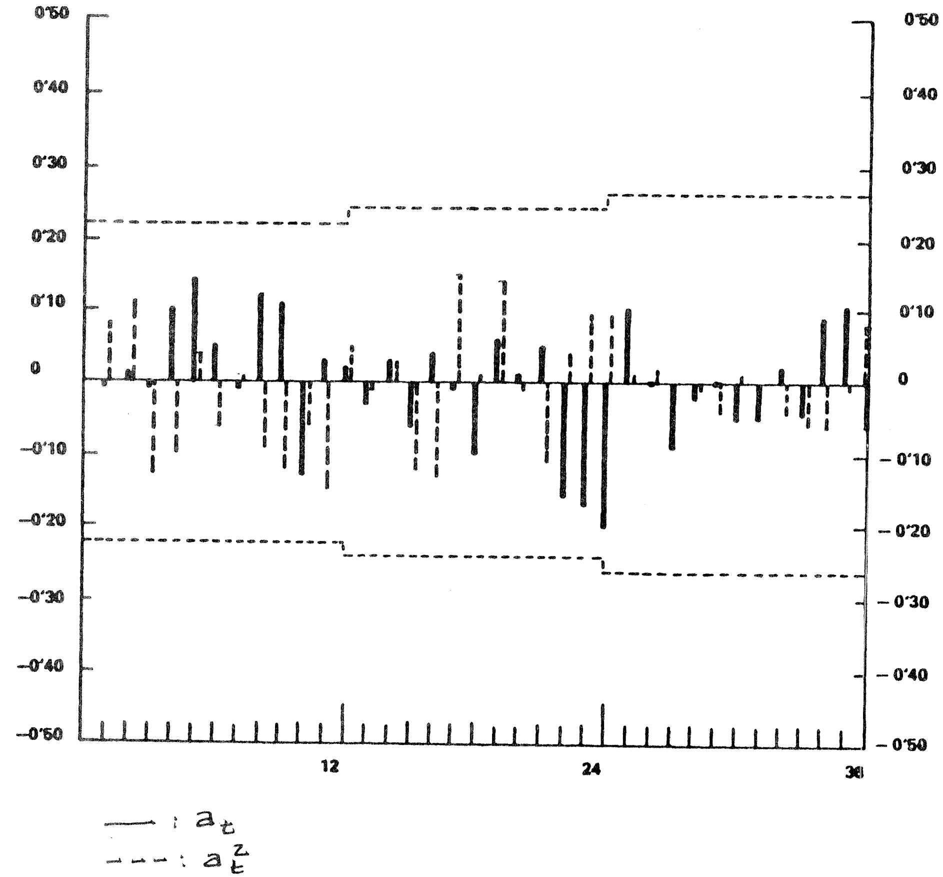


Figure 10

ACF of (a_t) and (a_t^2): Daily Series

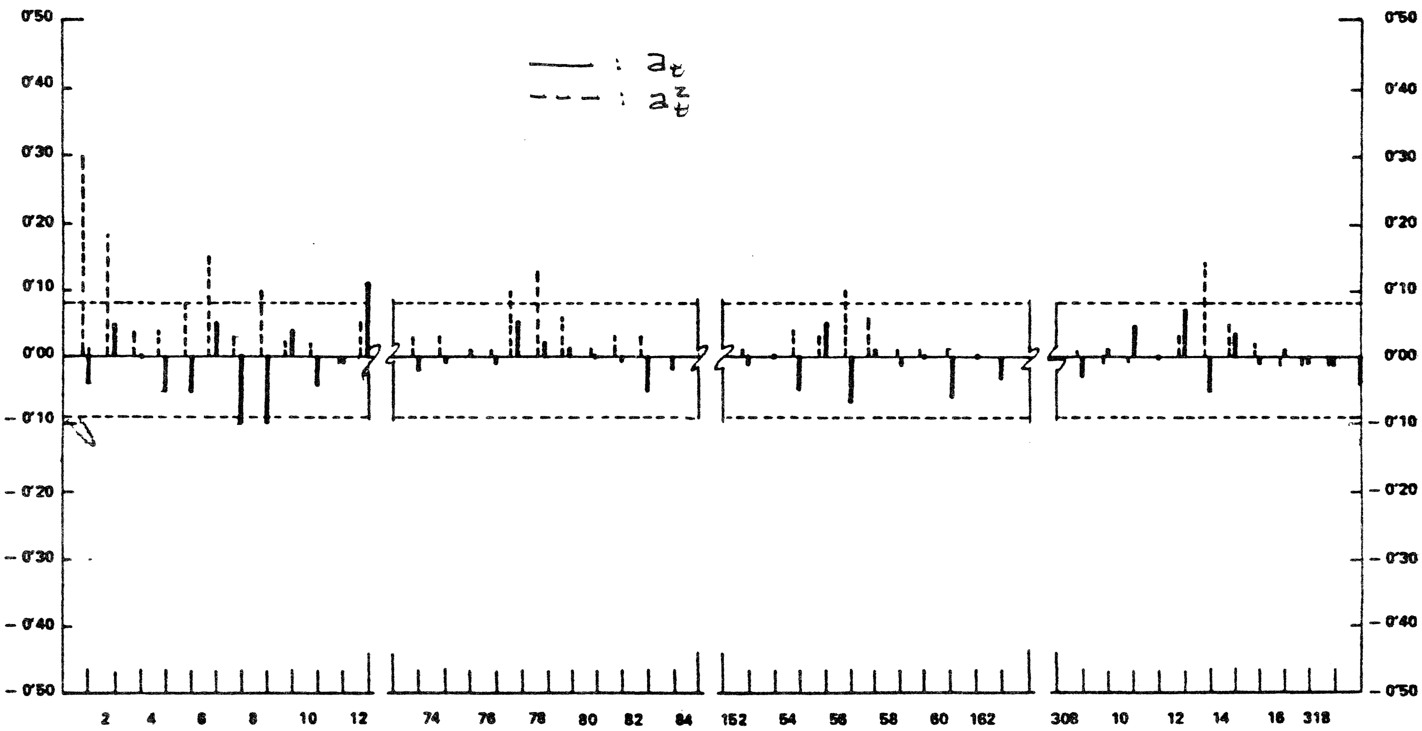


Figure 11

ACF of a_t^2: Model (3)

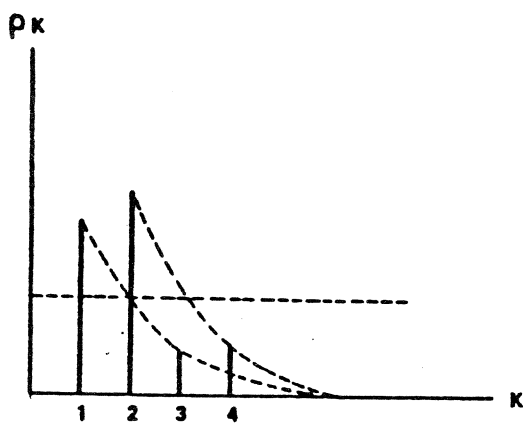


Figure 12

Contours of the Sum of Squares: Model (6)

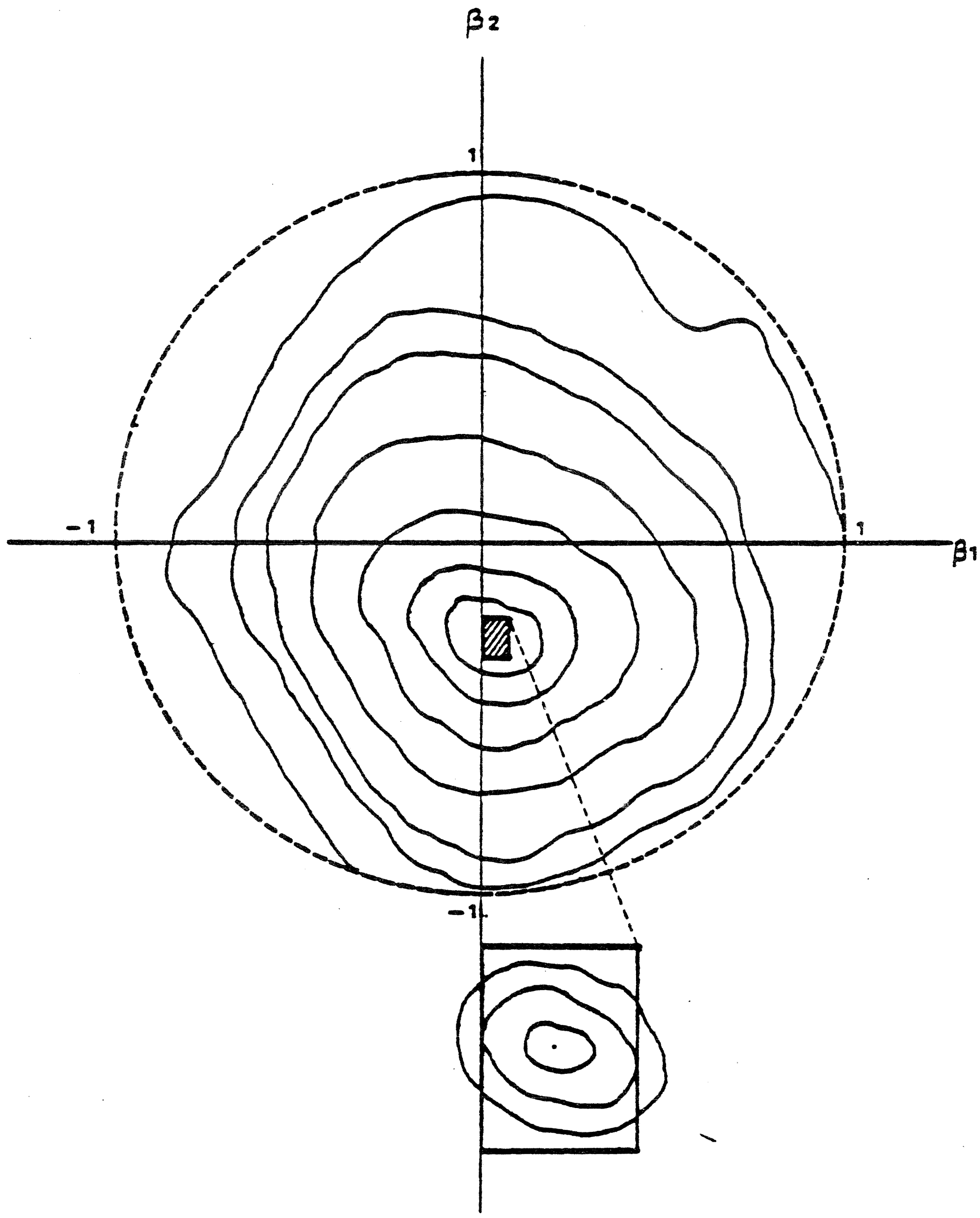


Figure 13

ACF of Bilinear Residuals (ε_t)

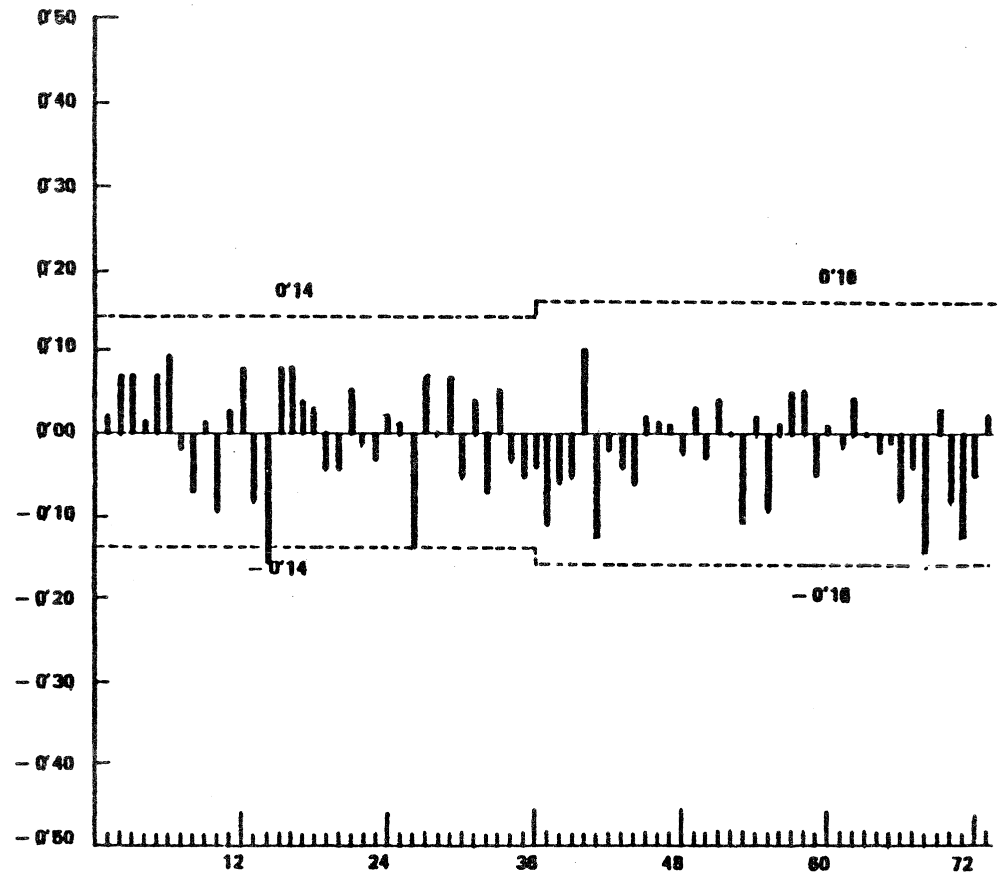


Figure 14

ACF of (ε_t^2)

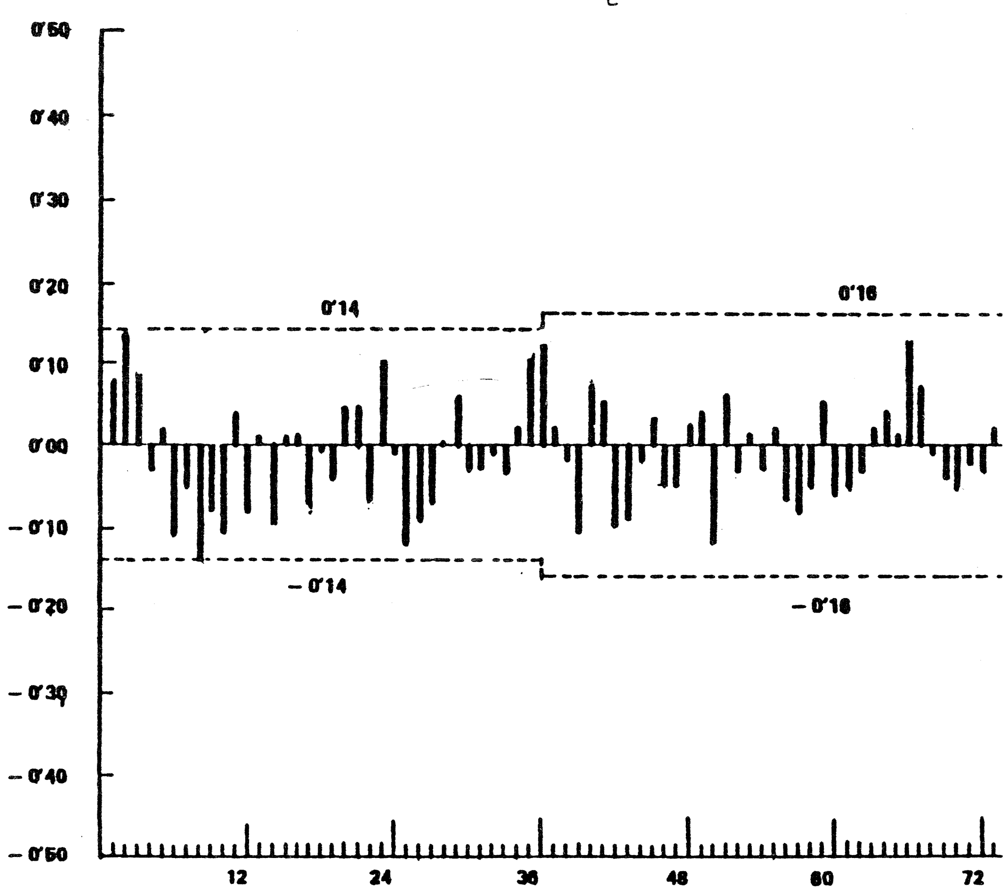


Figure 15

Forecast Errors

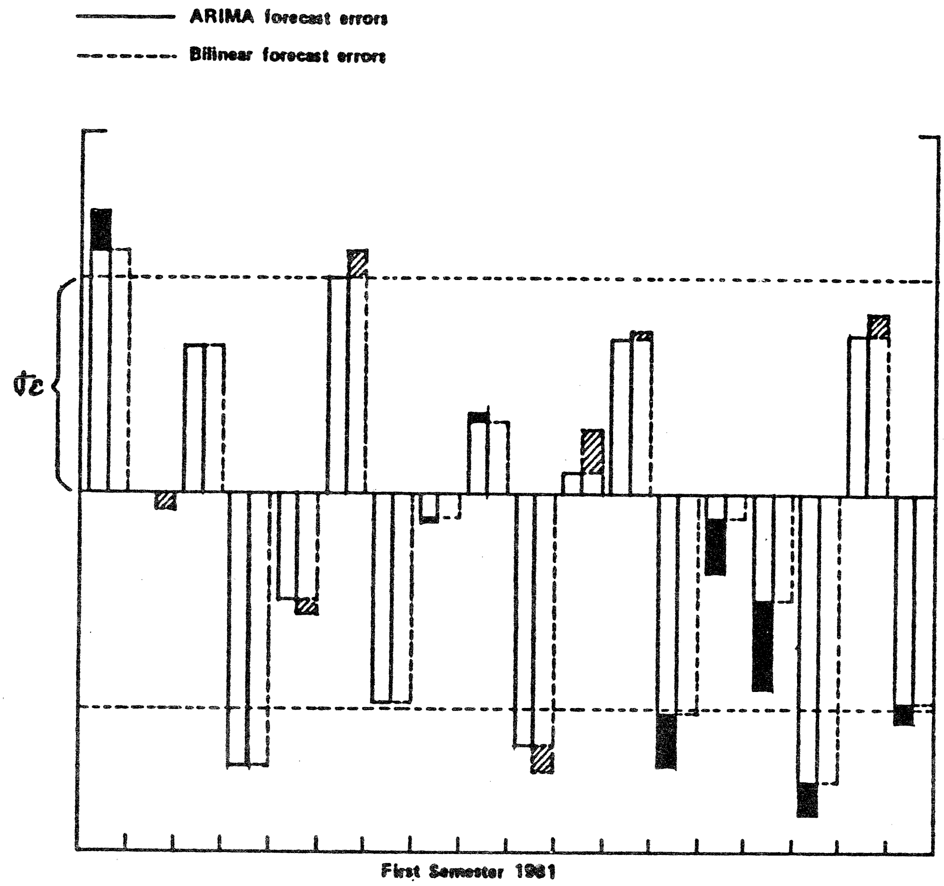


Figure 16

Forecast Errors: Changes in Linear Specification

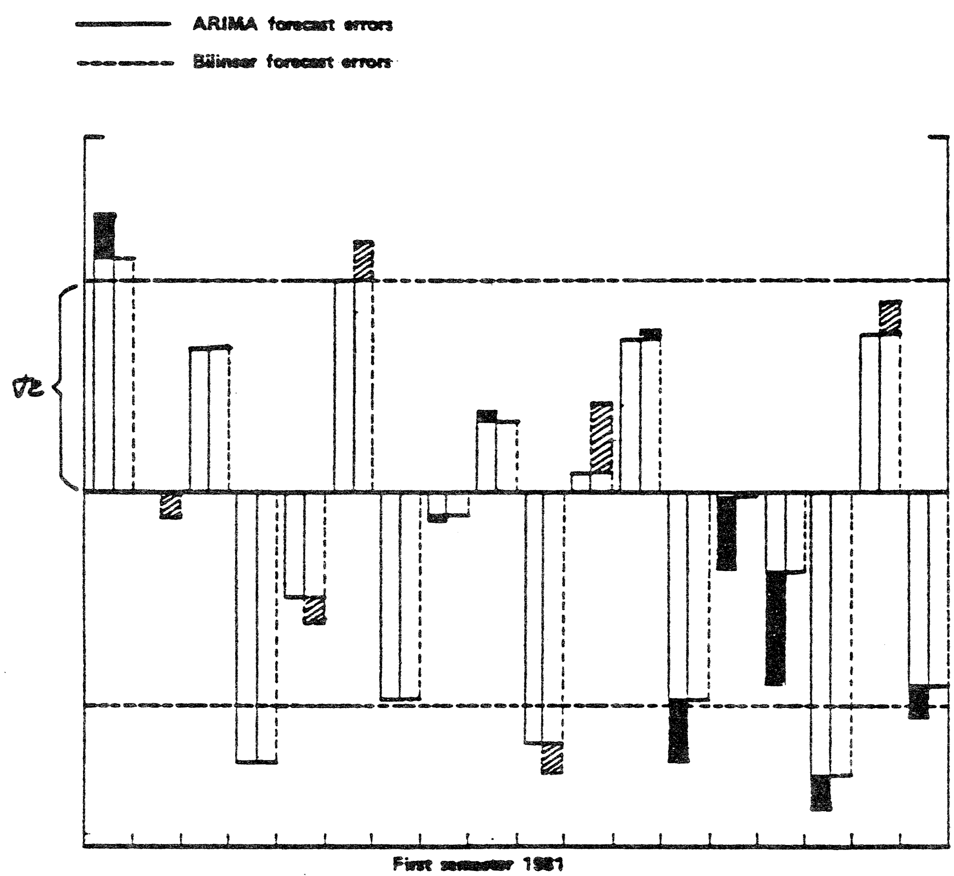


Figure 17

Contours of the Sum of Squares: Model (10a)

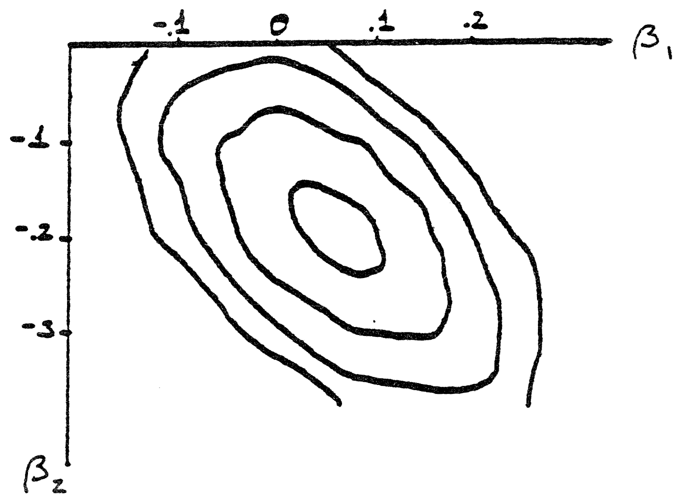
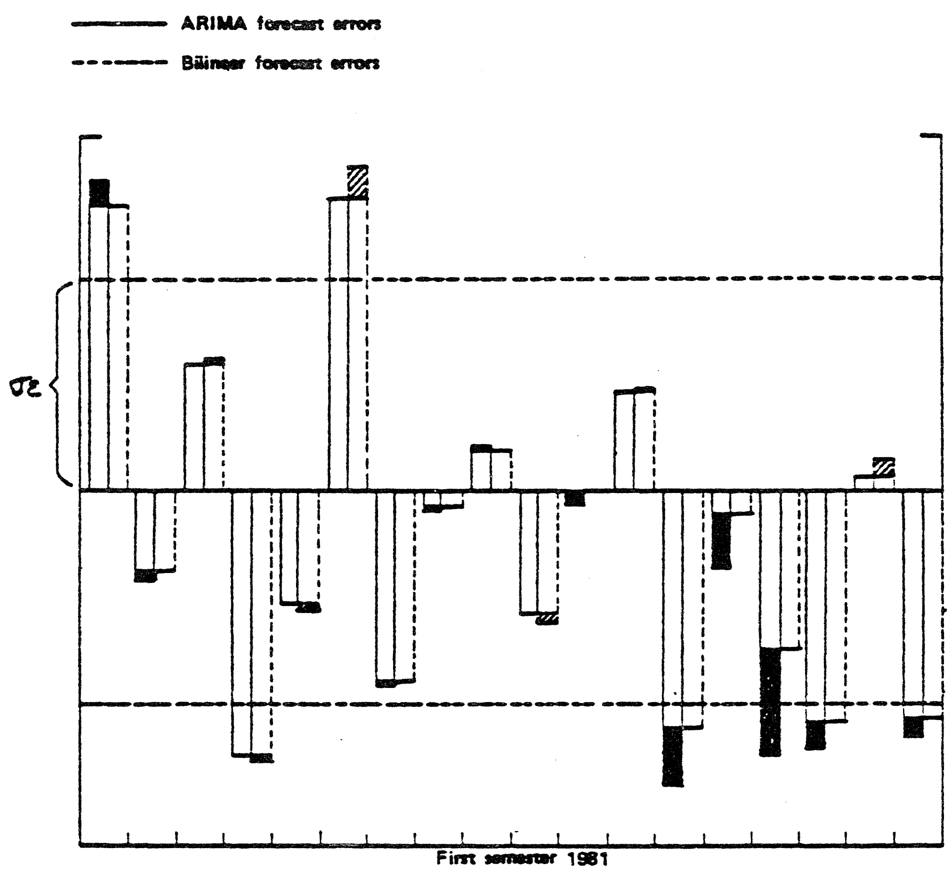


Figure 18

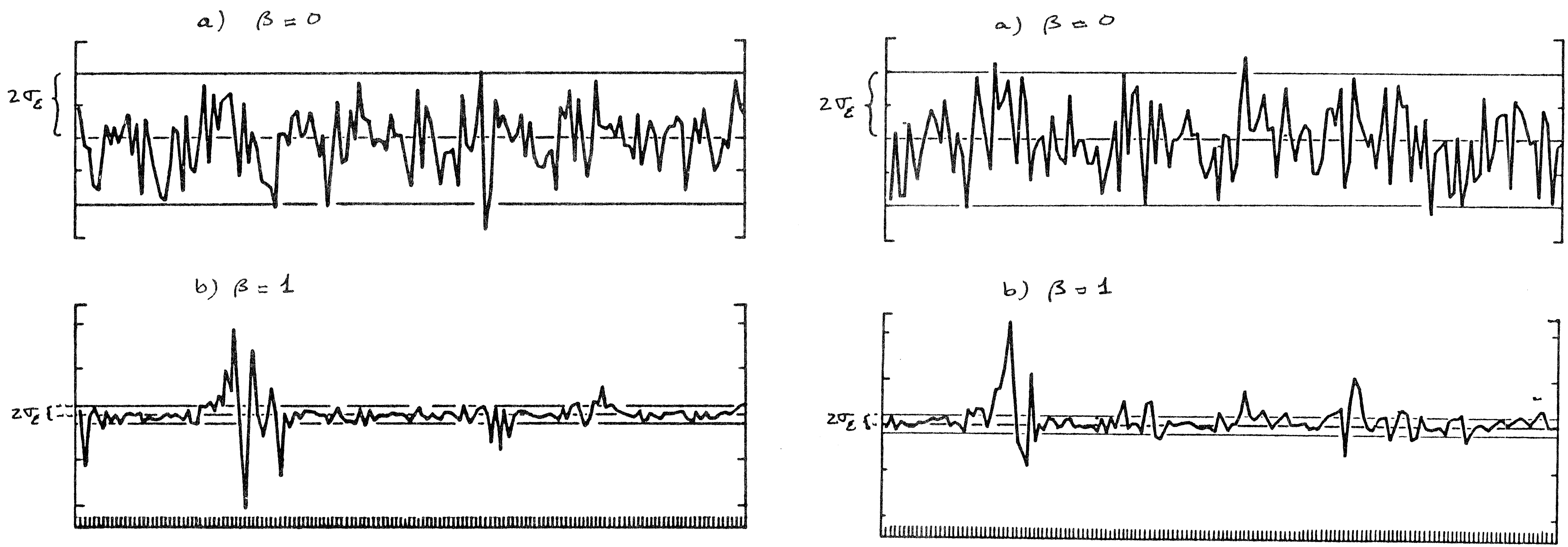
Forecast Errors: Changes in Bilinear Specification



Series Generated by $z_t = \beta z_{t-1}\, \varepsilon_{t-1} + \varepsilon_t$

A. First Random Sample

B. Second Random Sample



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